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Integration by Parts
Trigonometric Integrals

Integration by Parts
Integration by parts is a technique based on the product rule. U-substitution also involves products $g'(x)f\big(g(x)\big)$, but integration by parts is for products that are not related to each other: $f(x)g(x)$.

One example of the difference:

U-sub: $\int xe^{x^2}dx$ in this case, the derivative of x^2 is 2x, and the x out front is just a scalar multiple of that.

By parts: $\int xe^x\,dx$ in this case, the x out front is not related to the derivative of a composite function.

$$(uv)' = u'v + v'u$$

$$\int (uv)' = \int (u'v + v'u)$$

$$uv = \int u'v + \int v'u$$

$$\int udv = uv - \int vdu$$

Our task will be to choose u and dv functions so that the resulting new integral is easier to integrate.

LIATE
-Logs
-Inverse Trig Functions
-Algebraic
-Trig
-Exponential

When choosing the u function start at the top and work your way down.
When choosing dv start at the bottom.

Log and Inverse trig functions do not have integrals for these stand-alone functions.
But, we do have derivative rules for them.

Algebraic functions separate out into at least two subcategories:
-polynomials
-rational or radical functions

The polynomials are better for u since they will disappear if you take the derivative enough times.

Trig functions they will change but they won't disappear when you integrate or take the derivative.

The exponential function doesn't really change at all, but it won't necessarily become more complicated when you integrate.

Example.

$$\int xe^x dx$$

Choose u and dv: $u = x, dv = e^x dx$

$$du = dx, v = \int dv = \int e^x dx = e^x$$

$$\int udv = uv - \int vdu$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Example.

$$\int x^2 \sin x \, dx =$$

$$u = x^2, dv = \sin x \, dx$$
$$du = 2xdx, v = -\cos x$$

$$-x^2 \cos x - \int -2x \cos x \, dx = -x^2 \cos x + \int 2x \cos x \, dx =$$

$$u = 2x, dv = \cos x \, dx$$
$$du = 2dx, v = \sin x$$

$$-x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Example.

$$\int \arcsin x \, dx$$

$$u = \arcsin x \, , dv = dx$$
$$du = \frac{1}{\sqrt{1 - x^2}} dx, v = x$$

$$x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

This second integral I can integrate by substitution.

$$w = 1 - x^2, dw = -2xdx \rightarrow -\frac{1}{2}dw = xdx$$

$$\int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{1}{\sqrt{w}}\left(\frac{1}{2}\right)dw = -\frac{1}{2}\int w^{-\frac{1}{2}}dw = -\frac{1}{2}\left(2w^{\frac{1}{2}}\right) = -\sqrt{w} = -\sqrt{1-x^2}$$

$$\int \arcsin x\, dx = x \arcsin x + \sqrt{1-x^2} + C$$

Not all combinations of functions will work:

$$\int e^x \ln x\, dx$$

$$u = \ln x\,, dv = e^x$$
$$du = \frac{1}{x}dx, v = e^x$$

$$e^x \ln x - \int \frac{e^x}{x}dx$$

We still can't integrate this. If I continue, I will either get back to where I started, or it will get worse.

Example.

$$\int x^3 e^{x^2} dx$$

Normal guess:

$$u = x^3 dv = e^{x^2}dx$$

This isn't going to work because we can't integrate e^{x^2} by itself.

$$u = x^2, dv = xe^{x^2}dx$$
$$du = 2xdx, v = \frac{1}{2}e^{x^2}$$

We obtained this by substitution:

$$\int xe^{x^2}dx$$
$$w = x^2, dw = 2xdx \rightarrow \frac{1}{2}dw = xdx$$

$$\int \frac{1}{2}e^w dw = \frac{1}{2}e^w = \frac{1}{2}e^{x^2}$$

By parts formula:

$$\frac{1}{2}x^2e^{x^2} - \int 2x\left(\frac{1}{2}e^{x^2}\right)dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Example:

$$\int e^x \sin x \, dx =$$

$$u = \sin x \, , dv = e^x dx$$
$$du = \cos x \, dx, v = e^x$$

$$e^x \sin x - \int e^x \cos x \, dx$$

$$u = \cos x \, , dv = e^x dx$$
$$du = -\sin x \, dx, v = e^x$$

$$\int e^x \sin x \, dx = e^x \sin x - \left[e^x \cos x - \int -e^x \sin x \, dx \right]$$
$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Add the integral (on the right) to both sides of the equation:

$$\int e^x \sin x \, dx + \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2}[e^x \sin x - e^x \cos x] + C$$

Odd powers of secant also work this way.

Tabular Method

Is a method of doing several steps of integration by parts quickly, it works in certain cases: typically the u function is a polynomial, and the dv function has to be integrable alone (no substitutions)

Examples that work with method:

$$\int x^5 e^{3x} dx \, , \int x^4 \cos x \, dx \, , \int (x^3 + 2x - 1)\sqrt{x + 2} dx$$

sign	u	dv
+	x^5	e^{3x}
-	$5x^4$	$\frac{1}{3}e^{3x}$
+	$20x^3$	$\frac{1}{9}e^{3x}$
-	$60x^2$	$\frac{1}{27}e^{3x}$

+	$120x$	$\frac{1}{81}e^{3x}$
-	120	$\frac{1}{243}e^{3x}$
+	0	$\frac{1}{729}e^{3x}$

Alternating signs
Differentiate u until it goes to 0
Integrate dv

$$\int x^5 e^{3x} = \frac{1}{3}x^5e^{3x} - \frac{5}{9}x^4e^{3x} + \frac{20}{27}x^3e^{3x} - \frac{60}{81}x^2e^{3x} + \frac{120}{243}xe^{3x} - \frac{120}{729}e^{3x} + C$$

There are some problems that can be done with either integration by parts or change of variable:

$$\int x^2\sqrt{x+2}\,dx$$

For integration by parts: $u = x^2, dv = \sqrt{x+2} = (x+2)^{1/2}$

For change of variable: $u = \sqrt{x+2}, u^2 = x+2, x = u^2 - 2, dx = 2udu$

Trigonometric Integrals

In most cases, we are applying identities to set up for u-substitution or applying another identity. Pythagorean identities, or power reducing identities.

$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

When working with cosine and sine functions only:
1) Odd power of sine
2) Odd power of cosine
3) Even powers of both sine and cosine

1) If there is an odd power of sine, pull out one copy to be du, and convert any remaining sines to cosines using the Pythagorean identity

$$\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x\,(\sin x\ \ dx) = \int (1 - \cos^2 x)\cos^2 x\,(\sin x\,dx)$$

$$u = \cos x\,, du = -\sin x\, dx$$

2) If there is an odd power of cosine, pull out one copy to be du, and convert any remaining cosines to sines using the Pythagorean identity

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x\, (\cos x\ \ dx) = \int \sin^2 x\, (1 - \sin^2 x)(\cos x\, dx)$$

$$u = \sin x\,, du = \cos x\, dx$$

If they are both odd, you have a choice. I recommend converting to sines because you don't have the extra negative to deal with.

3) If you have only even powers: then you need to apply the power reducing identities until you get all linear functions of cosines

$$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(1 + \cos 2x) dx =$$

$$\frac{1}{4}\int 1 - \cos^2 2x\, dx = \frac{1}{4}\int 1 - \frac{1}{2}(1 + \cos 4x) dx$$

From here, simplify, and then finish integrating.

Tangent and secant integrals:

1) Even secants
2) Odd tangents
3) Odd secants with even tangents (or none)

1) The derivative of tangent is secant-squared, so pull out a pair of secants, and convert the rest to tangents.

$$\int \sec^4 x \tan x\, dx = \int \sec^2 x \tan x\, (\sec^2 x\, dx) = \int (1 + \tan^2 x) \tan x\, (\sec^2 x\, dx)$$

$$u = \tan x\,, du = \sec^2 x\, dx$$

2) Pull out one secant and one tangent to be du, and then covert the remaining even tangents to secants.

$$\int \sec^3 x \tan^3 x\, dx = \int \sec^2 x \tan^2 x\, (\sec x \tan x\, dx) = \int \sec^2 x\, (\sec^2 x - 1)(\sec x \tan x\, dx)$$

$$u = \sec x\,, du = \sec x \tan x\, dx$$

3) This integral "loops" with integration by parts and is extremely unfun.

Look at the example for $\int \sec^3 x\, dx$

The cosecant and cotangent examples follow exactly the same rules as for tangent and secant.