9/5/2023

Centers of Mass (lamina)
Probability Applications (examples)
Exponential/Logs/Growth and Decay
Hyperbolic Trig functions (?)

Center of mass in two dimensions (bounded by functions of one variable), and a constant mass density ρ.

First, to calculate the total mass of the region we integrate the density between the two curves: this is equivalent to multiplying the area of the region by the mass density to obtain the total mass.

$$M = \rho \int_a^b [f(x) - g(x)]dx$$

To calculate the center mass, we also need to find the moment of mass in each direction (x-direction and the y-direction).

The moment of mass from the y-axis M_y the moment in the x-direction. The center itself is $\bar{x} = \frac{M_y}{M}$.
The moment of mass from the x-axis is M_x, the moment in the y-direction. The center is $\bar{y} = \frac{M_x}{M}$.

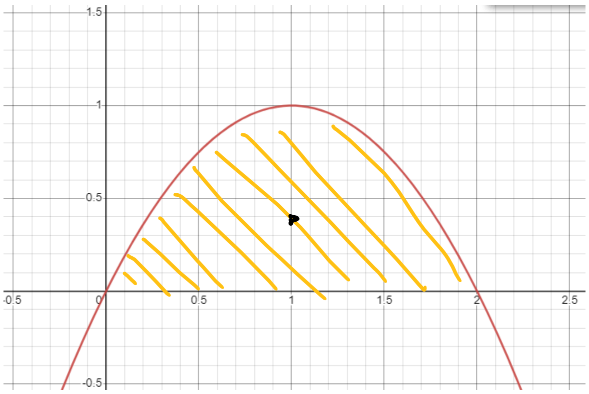
What is the moment of mass? In the discrete case, you have several masses set up along a seesaw (a balance), you want to know where to place the pivot point so that the masses on each side are balanced. The moment of mass is the sum of the masses x their distances from the zero. To get the pivot you divide by the sum of the masses to get the "average" distance.

The unit of moment of mass is distance times mass, and so dividing by the total mass gives you the average distance. The pivot point is like a weighted average.

For $M_y = \rho \int_a^b x[f(x) - g(x)]dx$

For $M_x = \rho \int_a^b \frac{1}{2}[f^2(x) - g^2(x)]dx$

Example.
Suppose we have a region bounded by the functions $y = 2x - x^2, y = 0$ with a constant mass density. Find the center of mass of the region.



interval is [0,2]

$$M = \rho\int_0^2 2x - x^2 dx = \rho\left[x^2 - \frac{1}{3}x^3\right]_0^2 = \rho\left[4 - \frac{8}{3}\right] = \frac{4\rho}{3}$$

$$M_y = \rho\int_0^2 x(2x - x^2)dx = \rho\int_0^2 2x^2 - x^3 dx = \rho\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \rho\left[\frac{16}{3} - 4\right] = \frac{4\rho}{3}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{4\rho}{3}}{\frac{4\rho}{3}} = 1$$

$$M_x = \frac{\rho}{2}\int_0^2 (2x - x^2)^2 dx = \frac{\rho}{2}\int_0^2 4x^2 - 4x^3 + x^4 dx = \frac{\rho}{2}\left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right]_0^2 =$$

$$\frac{\rho}{2}\left[\frac{32}{3} - 16 + \frac{32}{5}\right] = \frac{8\rho}{15}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{8\rho}{15}}{\frac{4\rho}{3}} = \frac{8\rho}{15}\times\frac{3}{4\rho} = \frac{2}{5}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{2}{5}\right)$$

How does this connect to the mean questions from the probability distributions?

Recall: $E(X) = \mu = \int_a^b xf(x)dx$
We don't need to divide by the area, because the area is 1.

Wrap up of applications and function types:

Exponential functions

$$\int e^x dx = e^x + C$$

$$\int e^{g(x)}g'(x)dx = \int e^u du = e^{g(x)} + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

Logarithmic function does not have a direct derivative: but it is derived from integration by parts

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \frac{g'(x)dx}{g(x)} = \ln|g(x)| + C$$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2}\ln|x^2+1| + C$$

A word of caution about log integrals: not all rational functions are log integrals.

$$\int \frac{1}{x^2+1} dx = \arctan x + C$$

$$\int \frac{x}{1+x^4} dx = \frac{1}{2}\arctan(x^2) + C$$

Growth and Decay

$$P(t) = P_0 e^{kt}$$

In growth problem, k is positive, and in decay problems k is negative.

The derivative is going to give the rate of change (in units of the population) at a point in time.

If they give the rate, then integrate to find the accumulation function.

Hyperbolic trig functions

$$\sin(x) = \frac{\left(e^{ix} - e^{-ix}\right)}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Compare to hyperbolic functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$$

$$e^{ix} + e^{-ix} = 2\cos(x)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$e^{ix} - e^{-ix} = 2i\sin(x)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

The hyperbolic trig functions are exactly the same, but the exponential functions are real.

$$\cos(0) = 1, \cosh(0) = 1$$
$$\sin(0) = 0, \sinh(0) = 0$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\frac{d}{dx}[\sinh(x)] = \cosh(x)$$

$$\frac{d}{dx}[\cosh(x)] = \sinh(x)$$

$$\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2 x$$

$$\frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}[\operatorname{sech}(x)] = -\operatorname{sech}(x)\tanh(x)$$

$$\frac{d}{dx}[\operatorname{csch}(x)] = -\operatorname{csch}(x)\coth(x)$$

Regular trig is based on the idea of a circle: $x^2 + y^2 = 1$

$$\cos^2 x + \sin^2 x = 1$$

Hyperbolic is based on the idea of a hyperbola: $x^2 - y^2 = 1$

$$\cosh^2(x) - \sinh^2(x) = 1$$

Brachistochrone problem, the shape of the wire hanging between two supports is a hyperbolic cosine function.

If you are given a ln(n) value with a hyperbolic trig function, expect that the function will evaluate to a fraction.