8/29/2023

Arc Length and Surface Area (of revolution)

See arc length handout for derivation details

Formula:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Some examples of functions that are easy to calculate with the arc length formula:

$$mx, x^{\frac{3}{2}}, \cosh(x), \ln(\cos x), cx^n + dx^{-m}$$

Find the length of arc of the function $f(x) = 2x^{\frac{3}{2}}$, on the interval [1,4].

First: find the derivative of the function. $f'(x) = 2\left(\frac{3}{2}\right)x^{\frac{1}{2}} = 3\sqrt{x}$

$$s = \int_1^4 \sqrt{1 + \left(3\sqrt{x}\right)^2} dx = \int_1^4 \sqrt{1 + 9x}\, dx$$

$$u = 1 + 9x, du = 9dx \to \frac{1}{9} du = dx$$

$$\int_{10}^{37} \frac{1}{9} u^{\frac{1}{2}} du = \frac{1}{9}\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_{10}^{37} = \frac{2}{27}\left[37^{\frac{3}{2}} - 10^{\frac{3}{2}}\right]$$

Example.

Find the length of arc on the function $f(x) = \cosh x$ on the interval $[0, \ln 2]$.

$$\cosh x = \frac{(e^x + e^{-x})}{2}$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$s = \int_0^{\ln 2} \sqrt{1 + (\sinh x)^2} dx = \int_0^{\ln 2} \sqrt{1 + \sinh^2 x}\, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x}\, dx = \int_0^{\ln 2} \cosh x\, dx =$$

$$\sinh x|_0^{\ln 2} = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - \frac{e^0 - e^0}{2} = \frac{1}{2}\left[2 - \frac{1}{2}\right] = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$$

Example.

Find the arc length of the function $f(x) = \frac{x^4}{4} + \frac{1}{8x^2} = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$ on the interval [1,2]

$$f'(x) = x^3 - \frac{1}{4}x^{-3}$$

$$s = \int_1^2 \sqrt{1 + \left(x^3 - \frac{1}{4}x^{-3}\right)^2}\, dx = \int_1^2 \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}}\, dx$$

$$\left(x^3 - \frac{1}{4}x^{-3}\right)\left(x^3 - \frac{1}{4}x^{-3}\right) = x^6 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}x^{-6} = x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}$$

$$\int_1^2 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16}x^{-6}}\, dx = \int_1^2 \sqrt{\left(x^3 + \frac{1}{4}x^{-3}\right)^2}\, dx = \int_1^2 x^3 + \frac{1}{4}x^{-3}\, dx = \frac{x^4}{4} - \frac{1}{8x^2}\Big|_1^2 =$$

$$\frac{16}{4} - \frac{1}{32} - \frac{1}{4} + \frac{1}{8} = \frac{123}{32}$$

Example.
Find the length of arc on the function $f(x) = e^x$ on the interval [0,1].

$$s = \int_0^1 \sqrt{1 + (e^x)^2}dx = \int_0^1 \sqrt{1 + e^{2x}}dx$$

Find the length of arc on the function $f(x) = x^3$ on the interval [0,1]

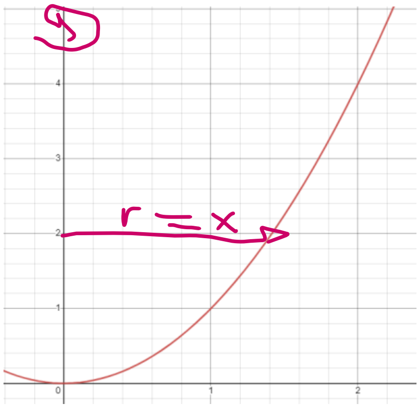
$$s = \int_0^1 \sqrt{1 + (3x^2)^2}dx = \int_0^1 \sqrt{1 + 9x^4}dx$$

We can evaluate these numerically.

Surfaces of revolution

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

Example.
Find the surface area of revolution for rotating the function $f(x) = x^2$ around the y-axis on the interval [0,2].

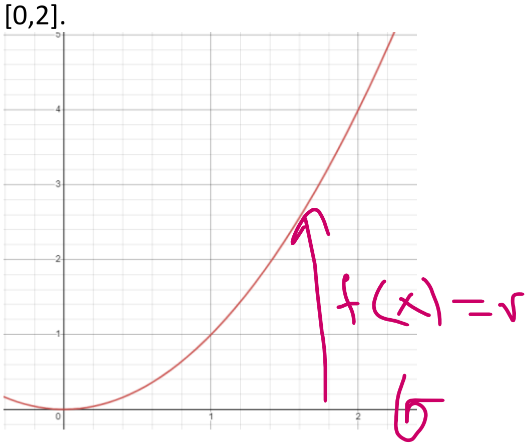


$$S = 2\pi \int_0^2 x\sqrt{1 + [2x]^2}dx = 2\pi \int_0^2 x\sqrt{1 + 4x^2}dx$$

$$u = 1 + 4x^2, du = 8xdx \rightarrow \frac{1}{8}du = xdx$$

$$S = 2\pi \int_1^{17} \frac{1}{8}u^{\frac{1}{2}}\, du = \frac{2\pi}{8}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{17} = \frac{\pi}{6}\left[17^{\frac{3}{2}} - 1\right]$$

Find the surface area of revolution for rotating the function $f(x) = x^2$ around the x-axis on the interval [0,2].



$$S = 2\pi \int_0^2 x^2\sqrt{1 + [2x]^2}dx = 2\pi \int_0^2 x^2\sqrt{1 + 4x^2}dx$$

This could be done by trig substitution, but we'll learn that next chapter. So for now, use technology to evaluate the integral.