11/9/2023

Separable Differential Equations
Logistic Equations/Autonomous Differential Equations

Separable differential equations:

$$\frac{dy}{dx} = f(x)g(y)$$

Homogenous differential equations can be made separable with a substitution (we'll leave for DiffEq)

For example:

$$x^2\frac{dy}{dx} = xy + y \rightarrow (x+1)y \rightarrow \frac{dy}{dx} = \left[\frac{x+1}{x^2}\right][y]$$

$$\frac{dy}{dx} = e^{x-y} \rightarrow [e^x][e^{-y}]$$

$$\frac{dy}{dx} = 2xy(y-1) \;\rightarrow [2x][y(y-1)]$$

The trick to solving these (functions that are the result of a chain rule or implicit differentiation) is to separate the y-variables to one side of the equation and the x-variables to the other.

$$\frac{dy}{dx} = \left[\frac{x+1}{x^2}\right][y]$$

Here: we multiply by dx and divide by y

$$\frac{1}{y}dy = \frac{x+1}{x^2}dx$$

$$\frac{1}{y}y' = \frac{x+1}{x^2}$$

$$\int \frac{1}{y}dy = \int \frac{x+1}{x^2}dx = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$\ln y = \ln x - \frac{1}{x} + C$$

It may be necessary to leave some equations like this in implicit form.

$$y = e^{\left(\ln x - \frac{1}{x} + C\right)} = e^{\ln x}e^{-\frac{1}{x}}e^{C} = Axe^{-\frac{1}{x}}$$

Example.

$$\frac{dy}{dx} = e^{x-y} = e^x e^{-y}$$

$$e^y dy = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C)$$

Logistic equation, population models, and autonomous differential equations, equilibria

Autonomous differential equations are equations that do not explicitly depend on the independent variable. They have only the derivatives and the original function.

$$\frac{dy}{dx} = f(y)$$

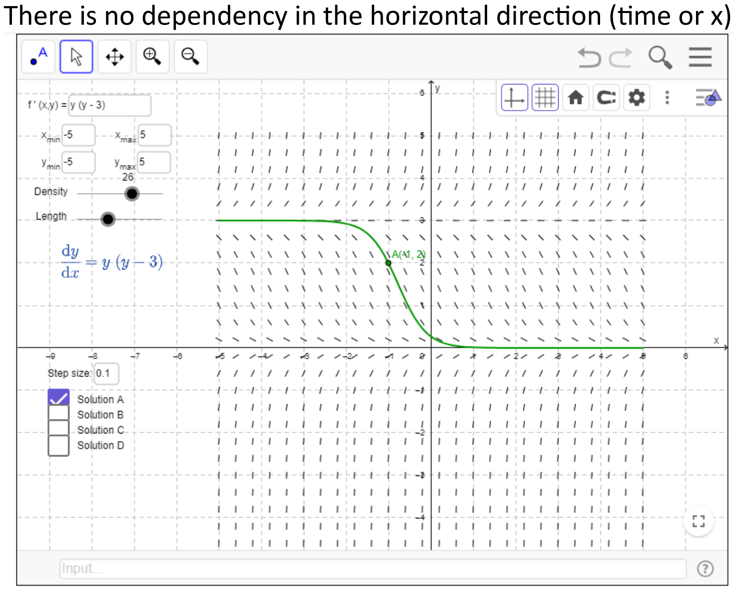
Typically these are time-based equations, and the slope of the solution does not depend on time (independent variable) only on y.

$$\frac{dy}{dx} = ky$$

Logistic equation: the function of y is a polynomial (in the simplest case, it's a quadratic)

$$\frac{dy}{dx} = ky(y - M) \quad or \quad \frac{dy}{dx} = ky\left(1 - \frac{y}{M}\right)$$

Direction fields:
There is no dependency in the horizontal direction (time or x) and so they are much easier to draw.



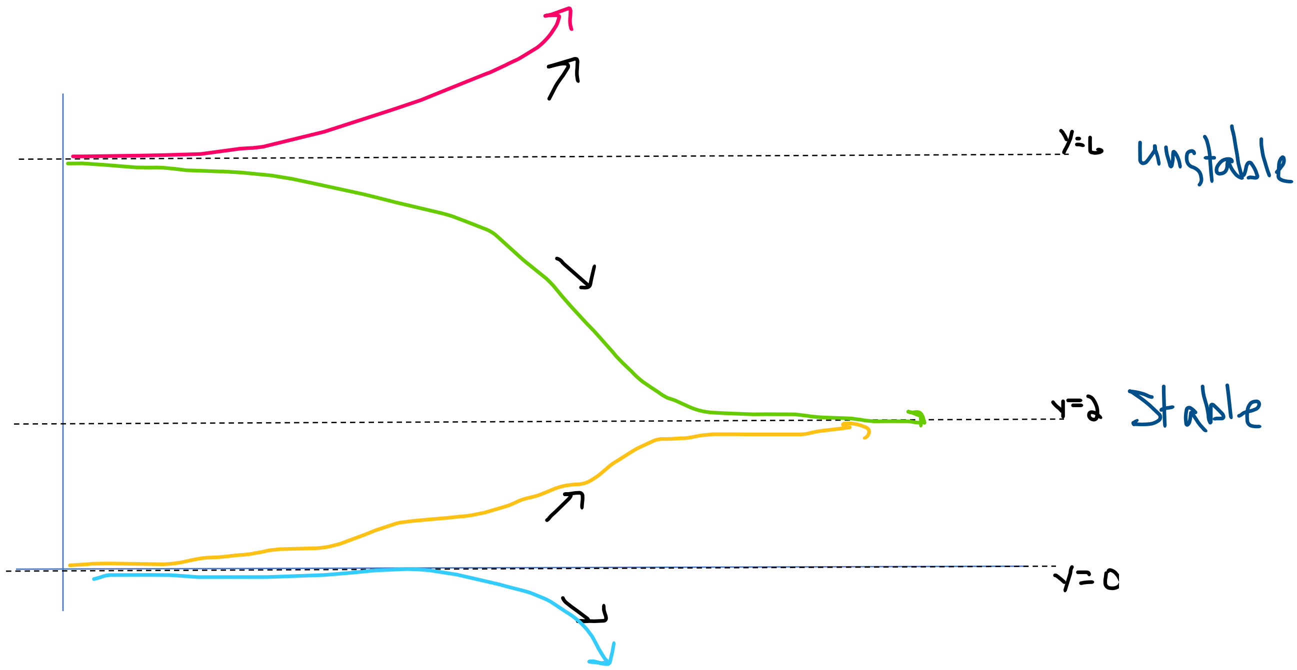
We can find the plot at a given value of y, and then the slope is the same all the way across horizontally.
For plotting these kinds of direction fields by hand:

1) Finding the equilibria (where is $\frac{dy}{dx} = 0$?)
2) Then check the signs of the slope in each region of the graph.

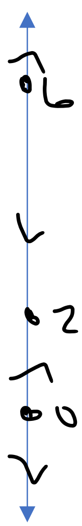
$$\frac{dy}{dx} = y(y-2)(y-6)$$

When is the slope 0?

$$y = 0, y = 2, y = 6$$

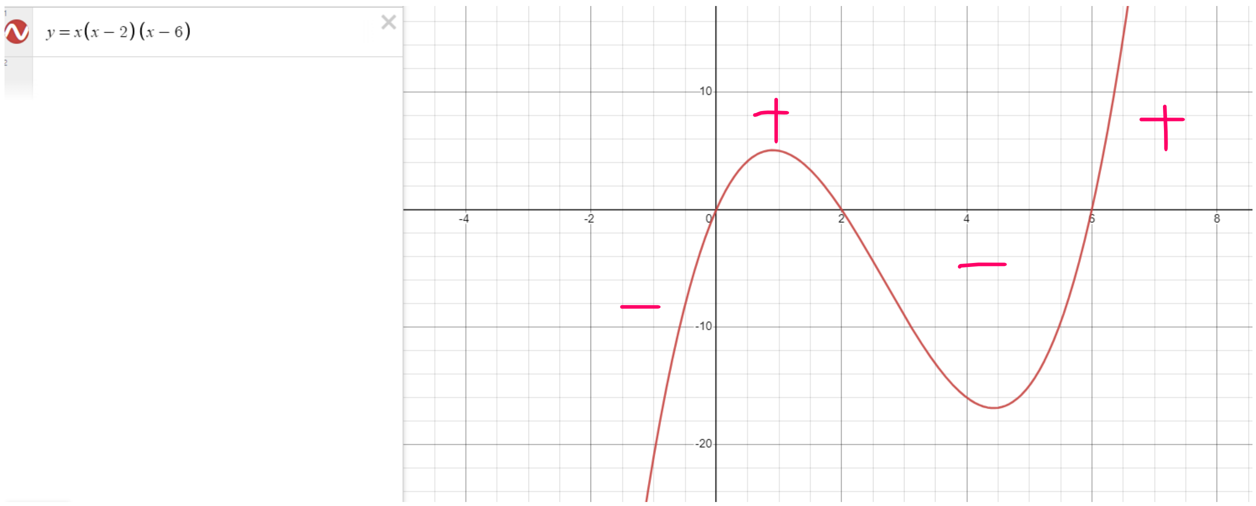


There is a related idea called a phase line



for this class, I won't ask for these. Use the direction field.

Phase plane (for an example like this):

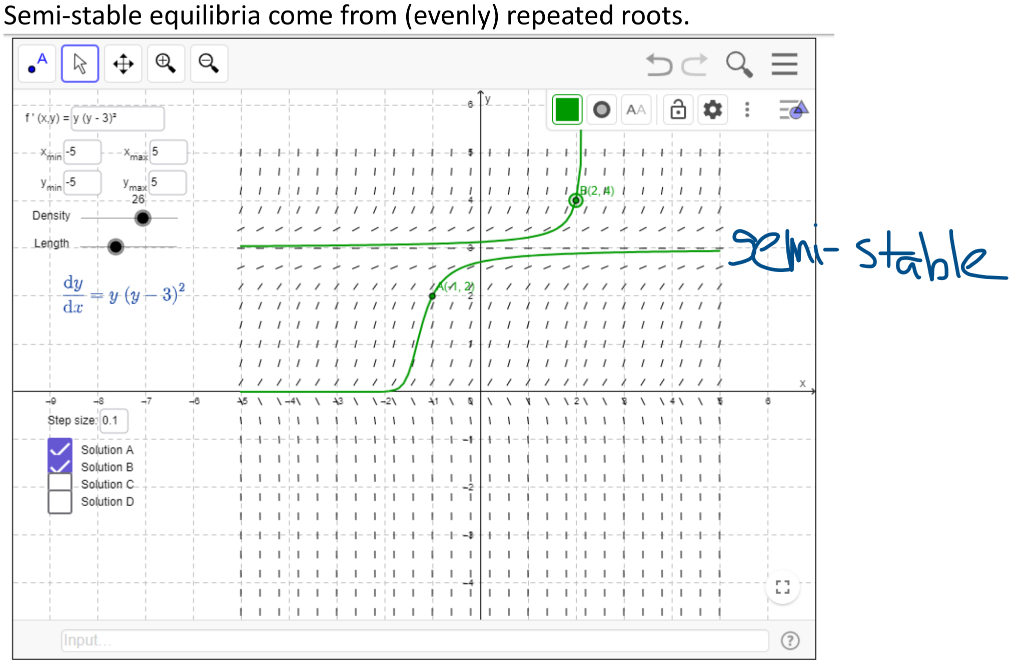


We can use this to determine the signs of the slopes. $\frac{dy}{dx}$ is the vertical axis (graphed as y), and the y is the horizontal axis (graphed as x).

Characterizing the equilibria:

1) Stable equilibrium: all the slopes point toward the equilibrium (positive below and negative above). Also called a carrying capacity.
2) Unstable equilibrium: all the slopes point away from the equilibrium (positive above, and negative below): threshold, or an extinction line
3) Semi-stable equilibrium: all the slopes point in the same direction on both sides (one side points toward the equilibrium and one side points away).

Semi-stable equilibria come from (evenly) repeated roots.



$$\frac{dy}{dx} = y(y-3)^2$$

Solving a logistic (differential) equation.

$$\frac{dy}{dt} = ky(1-y)$$

Because this is autonomous, this is a separable.

$$\frac{dy}{y(1-y)} = kdt$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{A - Ay + By}{y(1-y)}$$

$$A - Ay + By = 1$$

$$B - A = 0$$
$$A = 1, B = 1$$

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int kdt$$

$$\ln y - \ln(1-y) = kt + C$$

$$\ln\left(\frac{y}{1-y}\right) = kt + C$$

$$\frac{y}{1-y} = e^{kt+C} = e^{kt}e^{C} = Ae^{kt}$$

$$y = Ae^{kt}(1-y) = Ae^{kt} - Ae^{kt}y$$

$$y + Ae^{kt}y = Ae^{kt}$$

$$y\left(1 + Ae^{kt}\right) = Ae^{kt}$$

$$y = \frac{Ae^{kt}}{1 + Ae^{kt}}$$

$$y = \frac{Ae^{kt}}{1 + Ae^{kt}} \times \left(\frac{e^{-kt}}{e^{-kt}}\right) \rightarrow y = \frac{A}{e^{-kt} + A} = \frac{1}{Be^{-kt} + 1}$$