11/28/2023

Calculus in Polar Coordinates

Slope of a tangent line
Arc Length
Area of a polar graph, between polar curves

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Recall: $x(t) = r\cos(t)\,, y(t) = r\sin\,(t)$, where t can stand in for θ, and r can be $r(t)$.

$$x(t) = r(t)\cos(t)\,, y(t) = r(t)\sin(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{d(y(t))}{dt}\right)}{\frac{d(x(t))}{dt}} = \frac{\left(\frac{d}{dt}(r(t)\sin(t))\right)}{\frac{d}{dt}(r(t)\cos(t))} = \frac{(r'(t)\sin(t) + r(t)\cos(t))}{r'(t)\cos(t) - r(t)\sin(t)}$$

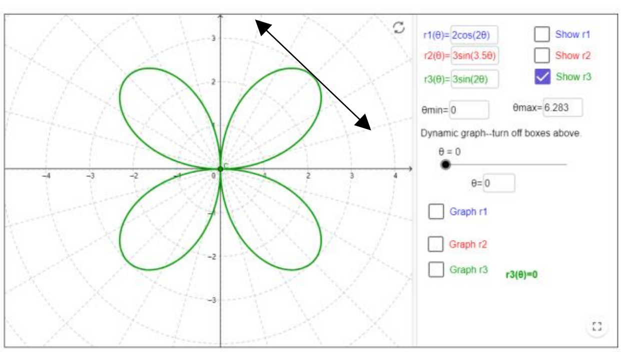
$$\frac{dy}{dx} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)}$$

Find the slope of the tangent line to polar graph $r = 3\sin(2\theta)$ at $\theta = \frac{\pi}{4}$

$$r'(\theta) = 6\cos(2\theta)$$

$$\frac{dy}{dx} = \frac{6\cos(2\theta)\sin(\theta) + 3\sin(2\theta)\cos\,(\theta)}{6\cos(2\theta)\cos(\theta) - 3\sin(2\theta)\,sin(\theta)}$$

$$= \frac{6\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + 3\sin\left(\frac{\pi}{2}\right)\cos\,(\frac{\pi}{4})}{6\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - 3\sin\left(\frac{\pi}{2}\right)sin(\frac{\pi}{4})} = \frac{6(0)\times\frac{1}{\sqrt{2}} + 3(1)\times\frac{1}{\sqrt{2}}}{6(0)\times\frac{1}{\sqrt{2}} - 3(1)\times\frac{1}{\sqrt{2}}} = \frac{3(1)\times\frac{1}{\sqrt{2}}}{-3(1)\times\frac{1}{\sqrt{2}}} = -1$$



Arc Length

$$s = \int_a^b \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} d\theta$$

You may need to apply identities to reduce the problem to something integrable: Pythagorean identities, power reducing identities, half-angle identities or double angle identities.

Example.
Find the length of arc of a quarter circle (between 0 and $\frac{\pi}{2}$) of radius 6.

$$r = 6$$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{(6)^2 + (0)^2} d\theta = \int_0^{\frac{\pi}{2}} 6 d\theta = 6\left(\frac{\pi}{2}\right) = 3\pi$$

The whole circumference is $C = 2\pi r = 2\pi(6) = 12\pi$, but only want a ¼ of the circle so 3π.

Example.
Find the circumference of the circle $r = 4\cos\theta$.

$$s = \int_0^{\pi} \sqrt{16\cos^2\theta + 16\sin^2\theta}\, d\theta = \int_0^{\pi} \sqrt{16(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^{\pi} 4 d\theta = 4\pi$$

Example.

$$r = 2 + 2\cos\theta$$

Find the arclength

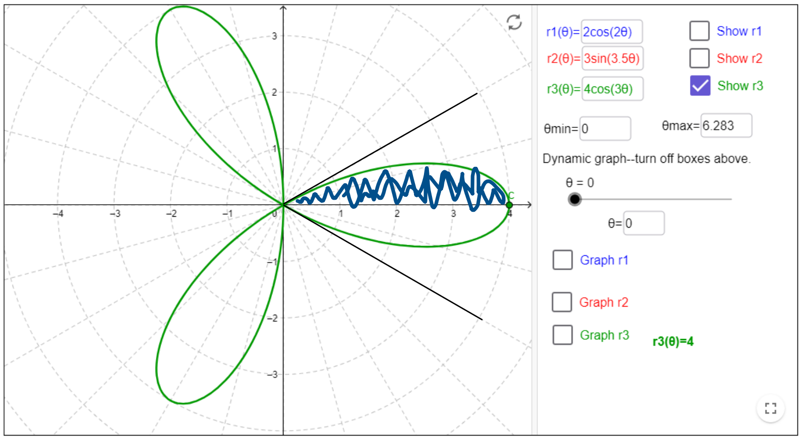
$$s = \int_0^{2\pi} \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2} d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta} = 2\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 4\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$4(2)\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 16\sin\left(\frac{\theta}{2}\right)\Big|_0^{\pi} = 16$$

Area of polar curve

Suppose I want to find the area of one petal of a rose $r = 4\cos 3\theta$



Limits of integration are where the curve intersects with the origin.

$$4\cos 3\theta = 0$$
$$\cos 3\theta = 0$$
$$\cos\alpha = 0$$

$$\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, etc. = 3\theta$$

$$\theta = \frac{\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{2}, etc.$$

The whole petal will be between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$, but we will worry about 0 to $\frac{\pi}{6}$ to find just the top half of the petal, and then we'll multiply by 2 for symmetry to get the rest.

Area formula for polar coordinates:

$$A = \frac{1}{2}\int_a^b [r(\theta)]^2 d\theta$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{6}} (4\cos 3\theta)^2 d\theta = \int_0^{\frac{\pi}{6}} 16\cos^2 3\theta\, d\theta = 16\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{6}} 1 + \cos(6\theta) d\theta =$$

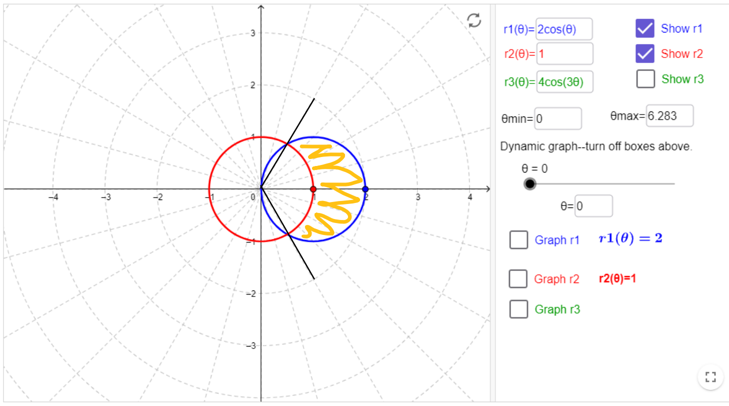
$$8\left[\theta + \frac{1}{6}\sin(6\theta)\right]_0^{\frac{\pi}{6}} = 8\left[\frac{\pi}{6}\right] = \frac{3\pi}{4}$$

Example.
What is the area of the circle $r = 4\cos\theta$?

$$A = \frac{1}{2}\int_0^{\pi} 16\cos^2\theta\, d\theta = 8\left(\frac{1}{2}\right)\int_0^{\pi} 1 + \cos 2\theta\, d\theta = 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = 4[\pi] = 4\pi$$

Radius of this circle is 2 (center is at 2, diameter is 4), $A = \pi r^2 = 4\pi$

Area between two polar curves.



What is the area inside the circle $r = 2\cos\theta$, but outside the circle $r = 1$.

Limits will be where the two circles intersect.

$$1 = 2\cos\theta$$

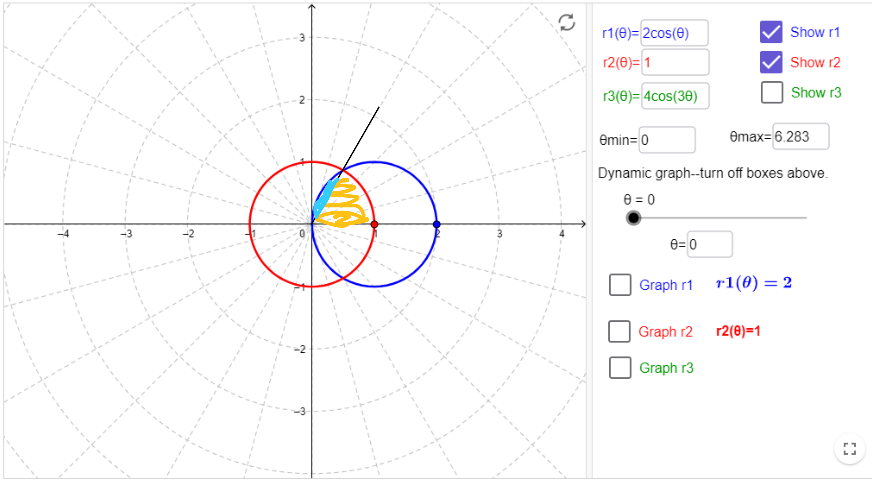
$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{3}} (4\cos\theta)^2 d\theta - 2\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{3}} (1)^2 d\theta =$$

$$\int_0^{\frac{\pi}{3}} 16\cos^2\theta - 1\, d\theta = \int_0^{\frac{\pi}{3}} 8(1+\cos 2\theta) - 1\, d\theta = \int_0^{\frac{\pi}{3}} 7 + 8\cos 2\theta\, d\theta =$$

$$7\theta + 4\sin 2\theta \Big|_0^{\frac{\pi}{3}} = \frac{7\pi}{3} + 4\sin\left(\frac{2\pi}{3}\right) = \frac{7\pi}{3} + 2\sqrt{3}$$



Find the area inside both $r = 1\ and\ r = 4\cos\theta$.

The intersections are at the same place as they were in the previous problems.
Integral 1: go from 0 to $\frac{\pi}{3}$ and the outer radius is $r = 1$.
Integral 2: got from $\frac{\pi}{3}$ to $\frac{\pi}{2}$ with the radius of $r = 4\cos\theta$.

$$A = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{3}}(1)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(4cos\theta)^2 d\theta\right]$$

Area of the inner loop of a limacon is also very common.