11/16/2023

Derivatives of vector-valued functions and parametric equations – first derivatives, second derivatives, vertical and horizontal tangents, concavity
Tangent vectors/tangent lines
Area under a parametric curve
Arc length
Surface area of a volume of solid of revolution in parametric form

We can represent functions or relations in x and y with a set of parametric equations as $x(t)$ and $y(t)$, or as a vector-valued function: $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Calculus on vector-valued functions:
Almost everything is done component by component.
If I want to find $\frac{d\vec{r}}{dt} = \vec{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \langle x'(t), y'(t) \rangle$

The derivative of a vector-valued function gives you the tangent vector to a curve. If you evaluate it at a point, it gives you the tangent vector at that point.

Example.

$$\vec{r}(t) = \langle t+2, t^2 - 1 \rangle$$
$$\vec{r}'(t) = \langle 1, 2t \rangle$$

If we want to find the tangent vector at some point, evaluate the derivative at that point.

$$\vec{r}'(1) = \langle 1, 2 \rangle$$

Slope is equivalent to the tangent: $\tan(\theta) = \frac{y}{x} \to \tan\theta = m = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$

The slope of the tangent line at the point t=1 is 2.
The slope of the tangent at any point t, for any function $\vec{r}$, is $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$. This is a relationship that also applied to parametric equations.

If I want to write the equation of the slope in regular cartesian form, then I find $\frac{dy}{dx}$ (the slope), and find the point the parametric curve is passing through, and then do the usual algebra.

$$\vec{r}(1) = \langle 3, 0 \rangle$$
$$y - 0 = 2(x - 3)$$
$$y = 2x - 6$$

But, I can also write the tangent in parametric or vector form:

$$T_{angent}(t) = t\langle x'(t), y'(t) \rangle + \langle x_0, y_0 \rangle = \langle \Delta x(t) + x_0, \Delta y(t) + y_0 \rangle$$

$$t\langle 1, 2 \rangle + \langle 3, 0 \rangle = \langle t + 3, 2t \rangle$$

$$x = t + 3, y = 2t$$

Derivatives of parametric functions:

$$x(t) = t^3, y(t) = t^2 - 1$$

Find $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dx}{dt} = x'(t) = 3t^2, \frac{dy}{dt} = y'(t) = 2t$$

$$\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$$

$$x = t^3 \to t = \sqrt[3]{x}$$

$$y(x) = \sqrt[3]{x^2} - 1 = x^{\frac{2}{3}} - 1$$

$$y'(x) = \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}} = \frac{2}{3t}$$

Second derivative:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d^2y}{dt^2}}{\left(\frac{dx}{dt}\right)^2} = \frac{1}{\frac{dx}{dt}} \times \frac{d}{dt}\left(\frac{dy}{dx}\right)$$

Example.

Starting from $x(t) = t^3, y(t) = t^2 - 1, \frac{dy}{dx} = \frac{2}{3t}$, find the second derivative, i.e. $\frac{d^2y}{dx^2}$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{2}{3}t^{-1}\right) = \frac{2}{3}(-1)t^{-2} = -\frac{2}{3t^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3t^2} \times \left(-\frac{2}{3t^2}\right) = -\frac{2}{9t^4}$$

Verify that this makes sense from the cartesian results.

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$$

$$\frac{d^2y}{dx^2} = y''(x) = \frac{2}{3}\left(-\frac{1}{3}\right)x^{-\frac{4}{3}} = -\frac{2}{9\sqrt[3]{x^4}} = -\frac{2}{9t^4}$$

Vertical and Horizontal tangents:
Horizontal tangents are when the slope of the tangent line is equal to 0.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

This implies that we get horizontal tangents when $\frac{dy}{dt} = 0$.

$$\vec{r}(t) = \langle t+2, t^2 - 1 \rangle$$

$$\vec{r}'(t) = \langle 1,2t \rangle$$

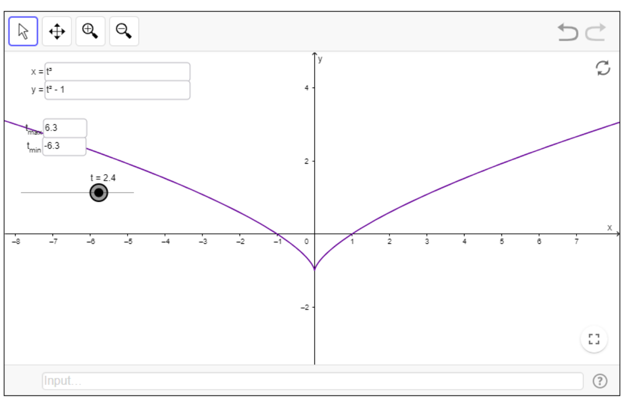
We will get a horizontal tangent when $2t = 0, or\ t = 0$.

Vertical tangents happen when the derivative $\frac{dy}{dx}$ is undefined, or $\frac{dx}{dt} = 0$

$$x(t) = t^3, y(t) = t^2 - 1, \frac{dy}{dx} = \frac{2}{3t}$$

$$\frac{dx}{dt} = 3t^2 \rightarrow 3t^2 = 0, t = 0$$

If the t had canceled in the denominator rather than the numerator, you'd have a hole in the derivative, and still could represent a point of vertical tangency or a cusp



You can also get vertical tangents when you have a vertical asymptote. Or if you have a relation like a circle.

Second derivatives can give us concavity. First derivatives can tell us where the graph is increasing or decreasing.

$$\frac{dy}{dx} = \frac{2}{3t}, \frac{d^2y}{dx^2} = -\frac{2}{9t^4}$$

Critical point at $t = 0$. When t>0, the derivative is also positive, so the function is increasing. When t<0, the derivative is negative, and so the function is decreasing.

For the second derivative, this is always negative. That means concave down. Where the second derivative is zero is a possible inflection point.

For what it's worth, limits, summations, integration, etc…. apply term-by-term to vector-valued functions.

Recall that the first derivative is like velocity, and so the magnitude of the velocity is speed: $\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$.

Area under a parametric curve.

In rectangular coordinates: $A = \int_a^b y(x)dx$
To parametric form:

$$A = \int_{t_0}^{t_1} y(t)\frac{dx}{dt}\,dt = \int_{t_0}^{t_1} y(t)x'(t)\,dt$$

Example. Find the area under the curve defined by the parametric equations $x(t) = t^3, y(t) = t^2 - 1$, on the interval [1,8] (these are values in t not x)

$$A = \int_1^8 (t^2 - 1)(3t^2)dt = \int_1^8 3t^4 - 3t^2 dt = \frac{3}{5}t^5 - t^3\Big|_1^8 = 19{,}660.8 - 512 - \frac{3}{5} + 1 = 19{,}149.2$$

Arc length of a parametric curve

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_a^b \|\vec{r}'(t)\|\,dt$$

$$\vec{r}(t) = \langle t + 2, t^2 - 1\rangle$$

Find the length of arc of the vector-valued function (parametric function) on the interval $[-1,2]$

$$s = \int_{-1}^{2} \sqrt{[1]^2 + [2t]^2}dt = \int_{-1}^{2} \sqrt{1 + 4t^2}\,dt \approx 6.1257\ldots$$

Example.

$$x(t) = t^3, y(t) = t^2 - 1$$

$$s = \int_a^b \sqrt{(3t^2)^2 + (2t)^2}dt = \int_a^b \sqrt{9t^4 + 4t^2}dt = \int_a^b \sqrt{t^2(9t^2 + 4)}dt = \int_a^b t\sqrt{9t^2 + 4}dt$$

This I could evaluate with regular u-sub and not need trig sub.

Surface of revolution

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

In parametric form:

$$S = 2\pi \int_a^b R(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = 2\pi \int_a^b R(t)\|\vec{r}'(t)\|dt$$

Recall that when we rotated around the x-axis, the $r(x)$ function was $y(x)$, but if we rotate around the y-axis, then the $r(x) = x$.

So in parametric form, if we rotate around the x-axis, then $R(t) = y(t)$, but if we rotate around the y-axis, then the $R(t) = x(t)$.

Example.
Rotate the parametric equations $x(t) = t^3, y(t) = t^2 - 1$ around the x-axis. What is the surface area of revolution between $[1,3]$?

$$S = 2\pi \int_1^3 (t^2 - 1)\sqrt{(3t^2)^2 + (2t)^2}\,dt$$

What if I rotated the function around the y-axis? What is the surface area of revolution?

$$S = 2\pi \int_1^3 t^3\sqrt{(3t^2)^2 + (2t)^2}\,dt$$

You can integrate them numerically from here.

On Tuesday, we will start talking about polar coordinates.