10/24/2023

Taylor and Maclaurin Series

(in the last lecture, on the fourth line from the bottom, that should be $2a = \frac{x^2}{125}$, not just a)

The only difference between a Taylor series and a Maclaurin series is that Maclaurin series are centered at 0, and Taylor series can be centered anywhere in the domain.

Taylor series are an extension of linear approximations that we did in calc 1.

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$(x-a) = \Delta x$$

The Taylor series extends this to create better approximations with higher order derivatives

$$f(x) \approx$$
$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{IV}(a)}{4!}(x-a)^4 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Sometimes this is referred to as a Taylor polynomial (only an approximation if you stop at a fixed value for n), except for polynomials (for which the derivative will eventually be zero).

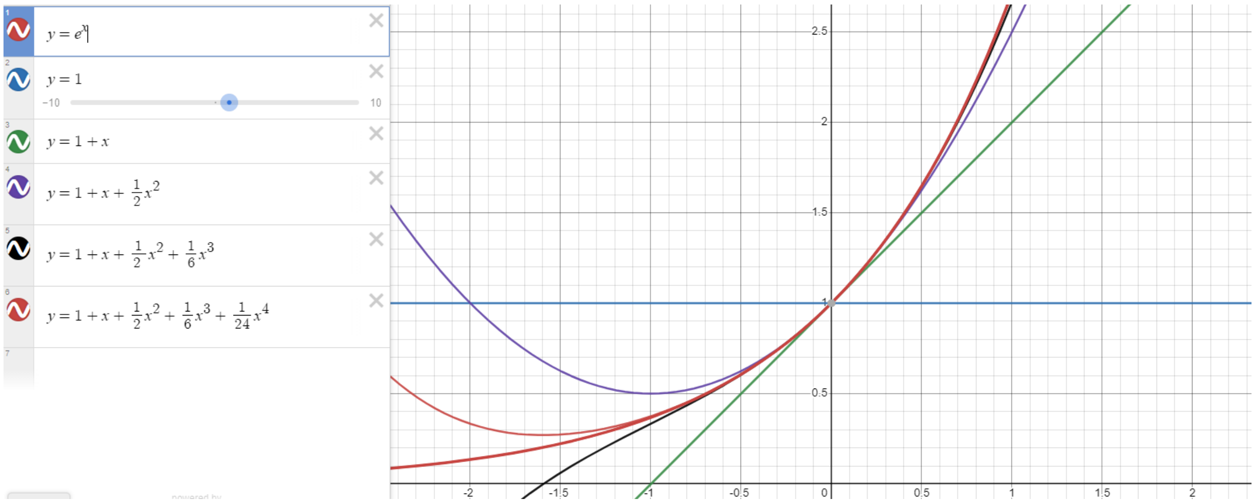
The power series is an infinite power series of this form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Find the Maclaurin series for the function $f(x) = e^x$.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$(x-a)^n$	$\frac{f^{(n)}(a)}{n!}(x-a)^n$
0	1	e^x	$e^0 = 1$	1	1
1	1	e^x	1	$x - a = x$	x
2	2	e^x	1	$(x-a)^2 = x^2$	$\frac{1}{2}x^2$
3	6	e^x	1	x^3	$\frac{1}{6}x^3$
4	24	e^x	1	x^4	$\frac{1}{24}x^4$
5	120	e^x	1	x^5	$\frac{1}{120}x^5$
6	720	e^x	1	x^6	$\frac{1}{720}x^6$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



Continuing with Taylor Series

Taking more terms from the series, produces better and better approximations to the original function for longer intervals of x, and infinite power series will converge to the original function wherever the power series converges.

If you can find a Taylor for a function at a point, then any other method that also produces a power series for the same function at the same point will produce the same power series.

Find a Taylor polynomial centered at π for $\sin(x)$, with 4 terms.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(a)$	$(x-a)^n$	$\frac{f^{(n)}(a)(x-a)^n}{n!}$
0	1	$\sin(x)$	0	1	0
1	1	$\cos(x)$	-1	$x-\pi$	$-(x-\pi)$
2	2	$-\sin(x)$	0	$(x-\pi)^2$	0
3	6	$-\cos(x)$	1	$(x-\pi)^3$	$\frac{(x-\pi)^3}{6}$
4	24	$\sin(x)$	0	$(x-\pi)^4$	0
5	120	$\cos(x)$	-1	$(x-\pi)^5$	$-\frac{(x-\pi)^5}{120}$
6	720	$-\sin(x)$	0	$(x-\pi)^6$	0
7	5040	$-\cos(x)$	1	$(x-\pi)^7$	$\frac{(x-\pi)^7}{5040}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}$$

$$P_7 = P_8 = -(x-\pi) + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)^5}{120} + \frac{(x-\pi)^7}{5040}$$

In general, you want at least 4 non-zero terms in order to establish a pattern from which to generate a general formula.

There is a table of common Maclaurin series in most calculus books. We will use the table for extending applications of Taylor series, but looking up a formula in the table should not replace showing work.

What can we do with these now that we have them?

Consider the exponential example from earlier:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1) We can use some of the Taylor series to calculate the value of convergent series.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

We can determine that this series converges using the ratio test. We can also determine the sum because this has the same form as the power series for e^x, our sum is e^2.

2) We can take derivatives and integrals of functions that we could not before and come up with a way of representing the resulting function

$$\int e^{x^2}\,dx$$

We can use the Taylor series generated for e^x and create a Taylor series for e^{x^2}.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$

3) We can create new series through multiplication, division, and use them to find limits.

Example with multiplication.

Find the first 5 terms of a power series for $\ln(1-x)\,e^x$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = (-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right)$$

$$\ln(1-x)\, e^x \approx (-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots)\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right)$$

$$\approx -x - x^2 - \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{1}{24}x^5 - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{12} - - \frac{x^3}{3} - \frac{x^4}{3} - \frac{x^5}{6} - \frac{x^4}{4} - \frac{x^5}{4} - \frac{x^5}{5} - \cdots$$

$$\approx -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5 - \cdots$$

Thursday we'll finish 6.4, also review for the second exam, which is on Halloween (a week from today).