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Root and Ratio Test
Series Tests overview

Root test: for an infinite defined by $\sum_{n=0}^{\infty} a_n$, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ the series converges, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, the series diverges. And if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the test is inconclusive.

Root test works best when the a_n already has something raised to nth power. (generally avoid with factorials)

The algebra is easier with the root test if you have n^n (or a similar form) in the expression.

Example.

$$\sum_{n=1}^{\infty} \frac{n^2 2^n}{(3n+1)^n}$$

Useful to know is that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$\sqrt{2} \approx 1.41 \ldots$$
$$\sqrt[3]{3} \approx 1.44..$$
$$\sqrt[10]{10} \approx 1.25 \ldots$$
$$\sqrt[100]{100} \approx 1.047 \ldots$$
$$\sqrt[10{,}000]{10{,}000} \approx 1.00092 \ldots$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2 2^n}{(3n+1)^n}} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{3n+1}\right)^n} = \lim_{n\to\infty} 1 \left(\frac{2}{3n+1}\right) = 0 < 1$$

This series converges.

Example.

$$\sum_{n=1}^{\infty} \left(\frac{4n+1}{3n-2}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+1}{3n-2}\right)^n} = \lim_{n\to\infty} \frac{4n+1}{3n-2} = \frac{4}{3} > 1$$

The series diverges

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Apply the root test:

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n^2}}=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n}}\right)^2=\frac{1}{\lim_{n\to\infty}\left(\sqrt[n]{n}\right)^2}=1$$

Similarly:

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

The root test: $\frac{1}{\lim_{n\to\infty}\sqrt[n]{n}}=1$

The root test is inconclusive for both of these series. In general, any rational expression will be inconclusive in the root (or ratio) test.

The ratio test:

For the series given by $\sum_{n=0}^{\infty}a_n$, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ the series converges, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, the series diverges, and if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ the test is inconclusive.

Example.

$$\sum_{n=1}^{\infty}\frac{n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{2^{n+1}}\right)}{\frac{n}{2^n}}\right|=\lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\times\frac{2^n}{n}\right|=\lim_{n\to\infty}\left|\frac{n+1}{2^n\cdot 2}\times\frac{2^n}{n}\right|=\lim_{n\to\infty}\frac{1}{2}\times\frac{n+1}{n}=\frac{1}{2}(1)=\frac{1}{2}<1$$

The series converges.

Example.

$$\sum_{n=0}^{\infty}\frac{4^n}{n!}$$

$$\lim_{n\to\infty}\left|\left(\frac{4^{n+1}}{(n+1)!}\right)\times\frac{n!}{4^n}\right|=\lim_{n\to\infty}\left|\left(\frac{4^n\cdot 4}{(n+1)n!}\right)\times\frac{n!}{4^n}\right|=\lim_{n\to\infty}\left|\left(\frac{4}{(n+1)}\right)\times\frac{1}{1}\right|=0<1$$

The series converges

$(2n)!$, $(2n+2)!=[2(n+1)]!$, $(3n!)!$, $(3n+3)!$

Example.

$$\sum_{n=1}^{\infty}\frac{n^2 3^n}{n^n}$$

$$\lim_{n\to\infty}\left|\frac{(n+1)^2 3^{n+1}}{(n+1)^{n+1}}\times\frac{n^n}{n^2 3^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 3^n\cdot 3}{(n+1)^n(n+1)}\times\frac{n^n}{n^2 3^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3}{(n+1)^n}\times\frac{n^n}{n^2}\right| =$$

$$3\lim_{n\to\infty}\frac{n+1}{n^2}\times\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = 3(0)\left(\frac{1}{e}\right) = 0 < 1$$

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

Recognize: this expression is the reciprocal of $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

The series converges

Sometimes it can be helpful to make a list of which things blow up faster than which other things.

$$\ln(n) < n < n^2 < 2^n < e^n < n! < n^n$$

Which blows up faster $n^n\ or\ (2n)!$?

To test $\lim_{n\to\infty}\frac{n^n}{(2n)!}$... if you get 0, then (2n)! blows up faster, if you get infinity, then n^n blows up faster, and if you get a constant, then they go at about the same rate.

Rewrite $0.\overline{46}$ as a fraction.

0.4646464646…=$\frac{46}{100}+\frac{46}{10000}+\frac{46}{10^6}+\frac{46}{10^8}+\cdots = 46\left(\frac{1}{10^2}+\frac{1}{10^4}+\frac{1}{10^6}+\frac{1}{10^8}+\cdots\right) 46\sum_{n=1}^{\infty}\left(\frac{1}{10^2}\right)^n =$ $\sum_{n=0}^{\infty} 46\left(\frac{1}{10^2}\right)^{n+1} = \sum_{n=0}^{\infty}\left(\frac{46}{100}\right)\left(\frac{1}{10^2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{46}{100}\right)\left(\frac{1}{10}\right)^{2n}$

$$\sum_{n=0}^{\infty}\left(\frac{46}{100}\right)\left(\frac{1}{10^2}\right)^n$$

Sum $= \frac{a}{1-r} = \left(\frac{46}{100}\right)\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{46}{100}\times\left(\frac{1}{\frac{99}{100}}\right) = \frac{46}{100}\times\frac{100}{99} = \frac{46}{99}$