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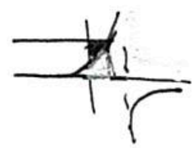
MTH 264, Exam #1, Fall 2023 Name KEY

**Instructions**: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the area bounded by the graphs $g(x) = \frac{4}{2-x}, y = 4, x = 0$. Sketch the region. (10 points)

$\frac{4}{2-x} = 4 \to \frac{1}{2-x} = 1 \to 2x = 1 \to x = 1$



$\int_0^1 4 - \frac{4}{2-x}\,dx = 4x + 4\ln|2-x|\Big|_0^1 = 4 + 4\cancel{\ln|1|}^{0} - 0 - 4\ln|2|$

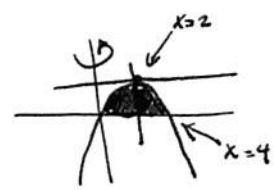
$= 4 - 4\ln 2$

2. Find the volume of the solid of revolution bounded by the graphs $y = \frac{1}{\sqrt{x+1}}, y = 0, x = 0, x = 4$, revolved around the $x$-axis using the disk or the washer method. (10 points)

$V = \pi\int_0^4 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx = \pi\int_0^4 \frac{1}{x+1}\,dx = \pi\ln|x+1|\Big|_0^4$

$\pi[\ln 5 - \cancel{\ln 1}^{0}] = \pi\ln 5$

3. Find the volume of the solid of revolution bounded by the graphs of $y = 4x - x^2, x = 0, y = 4$, and revolved around the $y$-axis, using the shell method. (10 points)




$V = 2\pi\int_0^2 x(4x-x^2)\,dx = 2\pi\int_0^2 4x^2 - x^3\,dx =$

$2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right|_0^2 = 2\pi\left(\frac{20}{3}\right) = \frac{40\pi}{3}$

or

$V = 2\pi\int_0^4 x(4x-x^2)\,dx = 2\pi\left(\frac{64}{3}\right) = \frac{128\pi}{3}$

4. Find the arc length of the graph $y = \ln(\sin x)$ on the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. (8 points)

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

$$S = \int_{\pi/4}^{3\pi/4} \sqrt{1 + \underbrace{\cot^2 x}_{\csc^2 x}}\, dx = \int_{\pi/4}^{3\pi/4} |\csc x|\, dx = \ln|\csc x + \cot x|\Big|_{\pi/4}^{3\pi/4} =$$

$$\approx 1.762747\ldots$$

5. On the interval [4,9], find the average value of the function $f(x) = \frac{1}{\sqrt{x}}$. (8 points)

$$\bar{f} = \frac{\int_4^9 \frac{1}{\sqrt{x}}\, dx}{9-4} = \frac{1}{5}\int_4^9 x^{-1/2}\, dx = \frac{1}{5} 2 x^{1/2}\Big|_4^9 =$$

$$\frac{2}{5}[3-2] = \frac{2}{5}$$

6. Find the surface area of the surface of revolution generated by revolving the curve $y = 1 - \frac{x^2}{4}$ over the interval [0,2] around the $x$-axis. (10 points)

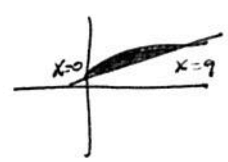
$$y' = -\frac{1}{4}\cdot 2x = -\frac{1}{2}x$$

$$S = 2\pi\int_0^2 \left(1 - \frac{x^2}{4}\right)\sqrt{1 + \left(-\tfrac{1}{2}x\right)^2}\, dx =$$

$$2\pi\int_0^2 (1 - x^2/4)\sqrt{1 + \tfrac{1}{4}x^2}\, dx =$$

$$2\pi[1.45527\ldots] \approx 9.143733\ldots$$

7. Find the centroid of a laminar sheet on constant density bounded by the graphs $y = \sqrt{x} + 1, y = \frac{1}{3}x + 1$. (16 points)




$$M = \int_0^9 (\sqrt{x}+1) - \left(\tfrac{1}{3}x + 1\right) dx = \int_0^9 \sqrt{x} - \tfrac{1}{3}x\, dx =$$

$$\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\Big|_0^9 = \frac{9}{2}$$

$$M_x = \frac{1}{2}\int_0^9 (\sqrt{x}+1)^2 - \left(\tfrac{1}{3}x+1\right)^2 dx = \frac{1}{2}\int_0^9 x + 2\sqrt{x} + \cancel{1} - \tfrac{1}{9}x^2 - \tfrac{2}{3}x - \cancel{1}\, dx = \frac{1}{2}\int_0^9 \tfrac{1}{3}x + 2\sqrt{x} - \tfrac{1}{9}x^2\, dx = \frac{45}{4}$$

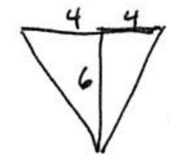
$$M_y = \int_0^9 x\left(\sqrt{x} - \tfrac{1}{3}x\right) dx = \int_0^9 x^{3/2} - \tfrac{1}{3}x^2\Big|_0^9 = \frac{81}{5}$$

$$\bar{X} = \frac{M_y}{M} = \frac{\cancel{81}^{9}}{5}\cdot\frac{2}{\cancel{9}} = \frac{18}{5}$$

$$\left(\frac{18}{5}, \frac{5}{2}\right)$$

$$\bar{Y} = \frac{M_x}{M} = \frac{\cancel{45}^{5}}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{\cancel{9}} = \frac{5}{2}$$

8. An open tank has the shape of a right circular cone (point at the bottom). The top of the tank is 8 feet across and 6 feet deep. How much work is done emptying the tank by pumping the water out over the top edge? Assume that the fluid in the tank is water and it has a weight-density of 62.4 lbs./ft$^3$. (10 points) $\cancel{V=\frac{1}{3}\pi r^2 h}$




$$W=\int_0^6 62.4\cdot(\pi)\left(\tfrac{2}{3}x\right)^2(6-x)\,dx$$

$$62.4\pi\,\tfrac{4}{9}\int_0^6 6x^2-x^3\,dx = 62.4\pi\cdot\tfrac{4}{9}\cdot 108$$

$$\approx 9409.698316\ldots$$

$\frac{4}{6}=\frac{r}{h}$

$4h=6r$

$r=\frac{2}{3}h$

Area of cylinder $\times h$

$\pi r^2 \times \Delta x$

9. For the following integrals, state which method you would use, and which basic integration rule. Do not actually perform the integration. Methods may include: substitution, change of variables, complete the square, add/subtract, trig identities, long division, partial fractions, by parts, trig substitution, etc. Basic integration rules may include: power rule, log rule, exponential rule, trig functions, inverse trig functions, etc. Some problems may require more than more method or rule. (5 points each)

a. $\int \frac{1}{x(\ln^3 x)}dx$ u-Sub $u=\ln x$, power rule

b. $\int x^3 \sin x\,dx$ int. by parts $u=x^3$, $dv=\sin x$, trig, power rule

c. $\int \arccos x\,dx$ int. by parts $u=\arccos x$, $dv=dx$ inv. trig, sub.

d. $\int \frac{\sqrt{16-x^2}}{x}dx$ trig sub trig, inv. trig

e. $\int \frac{1}{(x^2+2x+11)^{\frac{3}{2}}}dx$ trig sub, complete the square, trig, inv trig

f. $\int \frac{x}{16x^4-1}dx$ partial fractions, trig int, log rule

10. Use Trapezoidal Rule to approximate the area under the curve of $\int_1^2 \frac{\sin x}{x}dx$ for n=6. (10 points)

$h=\frac{2-1}{6}=\frac{1}{6}$

$\frac{1}{12}\left[f(1)+2f(1\tfrac{1}{6})+2f(\tfrac{8}{6})+2f(\tfrac{9}{6})+2f(\tfrac{10}{6})+2f(\tfrac{11}{6})+f(2)\right]=$

$0.659019$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

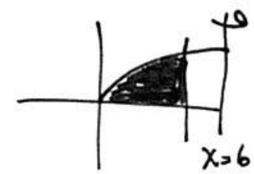
11. Use the definition of the hyperbolic tangent function to prove that its derivative is the hyperbolic secant function squared, i.e. that $\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$. (15 points)

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

12. Find the volume of the solid form from revolving the region formed by the equations $y = \sqrt{x}, y = 0, x = 4$ around the line x=6. Use the method of your choice. Sketch the region. (15 points)

$$V = 2\pi \int_0^4 (6-x)\sqrt{x}\, dx = 2\pi \int_0^4 6x^{1/2} - x^{3/2}\, dx$$

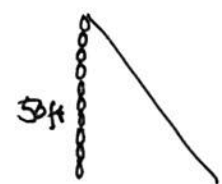
$$2\pi \left[ 6 \cdot \tfrac{2}{3} x^{3/2} - \tfrac{2}{5} x^{5/2} \right]_0^4 = 2\pi\left(\tfrac{96}{5}\right) = \frac{192\pi}{5}$$




13. Consider a 50-foot chain that weighs 4 pounds per foot hanging from a winch 50 feet above ground level. Find the work done by winding up the chain all the way. (20 points)

$$W = \int_0^{50} (50-x)4\, dx =$$

$$4\left[50x - \tfrac{1}{2}x^2\right]_0^{50} = 5000 \text{ ft-lbs}$$



14. Set up (but do not solve) this rational expression $\frac{x^5-7x^3+9x-15}{(x^2+3)(x-1)^3(x^2+1)^2(x+4)}$ for decomposition by partial fractions. (16 points)

$$\frac{Ax+B}{x^2+3}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}+\frac{Fx+G}{x^2+1}+\frac{Hx+I}{(x^2+1)^2}+\frac{J}{x+4}$$

15. Integrate by an appropriate method. (10 points each)

a. $\int e^{-3x}\sin 5x\,dx$

$u=\sin 5x \quad dv=e^{-3x}$

$du=5\cos 5x\,dx \quad v=-\frac{1}{3}e^{-3x}$

$-\frac{1}{3}e^{-3x}\sin 5x+\int\frac{5}{3}e^{-3x}\cos 5x\,dx$

$u=\cos 5x \quad dv=e^{-3x}$

$du=-5\sin 5x\,dx \quad v=-\frac{1}{3}e^{-3x}$

$-\frac{1}{3}e^{-3x}\sin 5x+\frac{5}{3}\left[-\frac{1}{3}e^{-3x}\cos 5x+\frac{-5}{3}\int e^{-3x}\sin 5x\,dx\right]$

$\int e^{-3x}\sin 5x\,dx=-\frac{1}{3}e^{-3x}\sin 5x+\frac{-5}{9}e^{-3x}\cos 5x-\frac{25}{9}\int e^{-3x}\sin 5x\,dx$

$\frac{34}{9}\int e^{-3x}\sin 5x\,dx=-\frac{1}{3}e^{-3x}\sin 5x+\frac{-5}{9}e^{-3x}\cos 5x+C$

$\rightarrow\int e^{-3x}\sin 5x\,dx=-\frac{3}{34}e^{-3x}\sin 5x+\frac{-5}{34}e^{-3x}\cos 5x+C$

b. $\int\frac{4x}{e^x}dx$

$=\int 4xe^{-x}\,dx$

$u=4x \quad dv=e^{-x}$

$du=4dx \quad v=-e^{-x}$

$-4xe^{-x}+\int 4e^{-x}\,dx=-4xe^{-x}-4e^{-x}+C$

c. $\int\sin^4 6\theta\,d\theta \Rightarrow \sin^2 6\theta\,\sin^2 6\theta$

$\int\frac{1}{4}(1-\cos 12\theta)(1-\cos 12\theta)\,d\theta=\frac{1}{4}\int 1-2\cos 12\theta+\cos^2 12\theta\,d\theta$

$=\frac{1}{4}\int 1-2\cos 12\theta+\frac{1}{2}(1+\cos 24\theta)\,d\theta=\frac{1}{4}\int\frac{3}{2}-2\cos 12\theta+\frac{1}{2}\cos 24\theta\,d\theta$

$=\frac{1}{4}\left[\frac{3}{2}\theta-\frac{1}{6}\sin 12\theta+\frac{1}{48}\sin 24\theta\right]+C$

d. $\int \cosh^4 2x\,dx$ $\qquad \cosh^2 x = \frac{\cosh 2x + 1}{2}$

$$\int \tfrac{1}{4}(1+\cos 4x)(1+\cosh 4x)\,dx = \tfrac{1}{4}\int 1 + 2\cosh 4x + \cosh^2 4x\,dx =$$

$$\tfrac{1}{4}\int 1 + 2\cosh 4x + \tfrac{1}{2} + \tfrac{1}{2}\cosh 8x\,dx = \tfrac{1}{4}\int \tfrac{3}{2} + 2\cosh 4x + \tfrac{1}{2}\cosh 8x\,dx$$

$$\tfrac{1}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\sinh 4x + \tfrac{1}{16}\sinh 16x\right] + C$$

e. $\int \frac{9x^3}{\sqrt{1+x^2}}\,dx$ $\qquad x = \tan\theta \qquad \sqrt{1+\tan^2 x} = \sec\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{9\tan^3\theta\,\sec^2\theta\,d\theta}{\sec\theta} = 9\int \tan^3\theta\,\sec\theta\,d\theta = 9\int(\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

$\tan^2\theta \cdot \tan\theta$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$ $\qquad 9\int u^2 - 1\,du = 9\left(\tfrac{1}{3}u^3 - u\right) + C = \tfrac{9}{3}\sec^3\theta - 9\sec\theta + C$

$$= \left[\tfrac{9}{3}(1+x^2)^{3/2} - \tfrac{9}{1}(1+x^2)^{1/2}\right] + C$$

f. $\int \frac{x^2 - x}{x^2+x+1}\,dx$

$$\begin{array}{r} 1 \\ x^2+x+1\;\overline{)\;x^2 - x} \\ -\;x^2 \mp x \mp 1 \\ \hline -2x - 1 \end{array}$$

$$\int 1 - \frac{2x+1}{x^2+x+1}\,dx = x - \ln|x^2+x+1| + C$$

$u = x^2+x+1$

$du = 2x+1\,dx$

16. Determine whether or not the integral converges or diverges. If the integral converges, state its value. (16 points)

$$\int_0^\infty \frac{1}{e^x + e^{-x}}\,dx \cdot \frac{e^x}{e^x} = \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$$

$u = e^x,\ du = e^x dx$

$$\int_1^\infty \frac{du}{u^2+1} = \lim_{b\to\infty} \arctan u \Big|_1^\infty = \lim_{b\to\infty} \arctan b - \arctan 1 =$$

$$\pi/2 - \pi/4 = \pi/4$$

Converges