MTH 267, Quiz #8, Fall 2022 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. A mass weighing 8 lbs stretches a spring 16 inches. If the mass is pushed upwards, contracting the spring a distance of 1 inch and then set in motion with a downward velocity of 4 ft/sec, and if there is no damping, and the system is driven by a force of 9cos(2t) pounds, set up the differential equation that models the system. ~~(7 points)~~

$$my'' + \gamma y' + ky = F(t)$$

$$\boxed{\begin{aligned} \tfrac{1}{4}y'' \quad + 6y &= 9\cos 2t \\ y'' + 24y &= 36\cos 2t \end{aligned}}$$

$F = ma$

$8 = m(32\ ft/sec^2)$

$\frac{1}{4} = m$

$y(0) = 1$

$y'(0) = -4$

$F = ky$

$8 = k\frac{16}{12}$

$k = \frac{8 \cdot 12}{16} = 6$

$y'' + 24y = 0 \qquad r^2 + 24 = 0$

$r = \pm\sqrt{24}\,i = \pm 2\sqrt{6}\,i$

$y_p = A\cos 2t + B\sin 2t$

$y_p' = -2A\sin 2t + 2B\cos 2t$

$y_p'' = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + 24A\cos 2t + 24B\sin 2t = 36\cos 2t$

$20A\cos 2t + 20B\sin 2t = 36\cos 2t \qquad B = 0$

$20A = 36$

$A = \frac{36}{20} = \frac{18}{10} = \frac{9}{5}$

$$\boxed{y(t) = c_1\cos(2\sqrt{6}\,t) + c_2\sin(2\sqrt{6}\,t) + \tfrac{9}{5}\cos 2t}$$

2. A series circuit has a capacitor of 10^{-5} F, a resistor of 500 Ω and an inductor of 0.7 H. The initial charge on the capacitor is 10^{-7} C and there is no initial current. Find the charge Q on the capacitor at any time t. What is the **charge** of the system at any time t? ~~(10 points)~~

$Lq'' + Rq' + \frac{1}{C}q = E(t)$

$q(0) = 10^{-7}$

$q'(0) = 0$

$0.7q'' + 500q' + \frac{1}{10^{-5}}q = 0$

$0.7r^2 + 500r + 10^5 = 0$

$r = \frac{-500 \pm \sqrt{500^2 - 4(0.7)(10^5)}}{2(0.7)} = \frac{-500 \pm \sqrt{-30{,}000}}{1.4} = \frac{-2500}{7} \pm \frac{500\sqrt{3}}{7}$

$q(t) = c_1 e^{-\frac{2500}{7}t}\cos\left(\frac{500\sqrt{3}}{7}t\right) + c_2 e^{-\frac{2500}{7}t}\sin\left(\frac{500\sqrt{3}}{7}t\right) \longrightarrow$

3. When does resonance occur? Explain. Provide an example.

when there is no damping and the forcing function and the natural frequency is similar in size but not identical such as if the natural frequency is 7, but the forcing function has frequency 8.

2. continued

$$q(t) = c_1 e^{-\frac{2500}{7}t}\cos\left(\frac{500\sqrt{3}}{7}t\right) + c_2 e^{-\frac{2500}{7}t}\sin\left(\frac{500\sqrt{3}}{7}t\right)$$

$$q'(t) = -\frac{2500}{7}c_1 e^{-\frac{2500}{7}t}\cos\left(\frac{500\sqrt{3}}{7}t\right) - \frac{500\sqrt{3}}{7}c_1 e^{-\frac{2500}{7}t}\sin\left(\frac{500\sqrt{3}}{7}t\right)$$
$$-\frac{2500}{7}c_2 e^{-\frac{2500}{7}t}\sin\left(\frac{500\sqrt{3}}{7}t\right) + c_2\frac{500\sqrt{3}}{7}e^{-\frac{2500}{7}t}\cos\left(\frac{500\sqrt{3}}{7}t\right)$$

$$q(0) = 10^{-7} = c_1(1)(1) + \cancel{c_2(1)(0)}^{0}$$

$$c_1 = 10^{-7}$$

$$q'(0) = 0 = -\frac{2500}{7}10^{-7}(1)(1) - \cancel{\frac{500\sqrt{3}}{7}10^{-7}(1)(0)}^{0} - \cancel{\frac{2500}{7}c_2(1)(0)}^{0} + c_2\frac{500\sqrt{3}}{7}(1)(1)$$

$$\frac{2500}{7}10^{-7} = c_2\frac{500\sqrt{3}}{7}$$

$$\frac{5\times10^{-7}}{\sqrt{3}} = c_2$$

$$q(t) = 10^{-7}e^{-\frac{2500}{7}t}\cos\left(\frac{500\sqrt{3}}{7}t\right) + \frac{5\times10^{-7}}{\sqrt{3}}e^{-\frac{2500}{7}t}\sin\left(\frac{500\sqrt{3}}{7}t\right)$$