MTH 267, Quiz #10, Fall 2022 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use Laplace transforms to solve $y'' - 2y' + 2y = \cos t, y(0) = 1, y'(0) = 0$.

$$s^2Y(s) - 2sY(s) + 2y(0) - sy(0) - y'(0) + 2Y(s) = \frac{s}{s^2+1}$$

$$Y(s)\left[s^2 - 2s + 2\right] + \cancel{2 - 2(1)} - 0 = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+1} \cdot \frac{1}{s^2-2s+2} = \frac{s}{s^2+1} \cdot \frac{1}{(s^2-2s+1)+1} = \frac{s}{(s^2+1)[(s-1)^2+1]} \rightarrow$$

2. Express each piecewise function in terms of the unit step function.

a. $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$

$$= t^2 + \begin{cases} 0, & 0 \le t < 2 \\ 1-t^2, & t \ge 2 \end{cases}$$

$t^2 +$ or $u_2(t)(1-t^2)$
$u(t-2)(1-t^2)$

$t^2 + u_2(t)(1-t^2)$ or
$t^2 + u(t-2)(1-t^2)$

b. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t-2, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

$$t + \begin{cases} 0 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ -2 & 2 \le t < 3 \\ -t & t \ge 3 \end{cases} \quad \begin{matrix} \downarrow -1 \\ \downarrow -1 \\ \downarrow -t+2 \end{matrix}$$

$t - u_1(t) - u_2(t) + (2-t)u_3(t)$ or
$t - u(t-1) - u(t-2) + (2-t)u(t-3)$

3. Find the inverse Laplace transform (using the table) of each function.

a. $F(s) = \frac{e^{-2s}}{s^2+s-2}$

$(s+2)(s-1)$

$$e^{-2s}\left(\frac{A}{s+2} + \frac{B}{s-1}\right)$$

$As - A + Bs + 2B = 1$
$A + B = 0$
$-A + 2B = 1$
$3B = 1$
$B = 1/3$
$A = -1/3$

$$\frac{1}{3}e^{-2s}\left(\frac{-1}{s+2} + \frac{1}{s-1}\right)$$

$$\frac{1}{3}u_2(t)\left[-e^{-2(t-2)} + e^{t-2}\right]$$

$$u_2(t) \cdot \left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right]$$

$$= \begin{cases} 0 & t \le 2 \\ -1/3\,e^{-2(t-2)} + \frac{1}{3}e^{t-2} & t > 2 \end{cases}$$

b. $F(s) = \frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$

$$\boxed{\int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau}$$

4. Use the table to find the Laplace transform of $f(t) = \int_0^t (t-\tau)\cos 2\tau \, d\tau$.

$t \quad \cos 2t$

$$\frac{1}{s^2}\left(\frac{s}{s^2+4}\right)$$

1. continued

$$Y(s) = \frac{s}{(s^2+1)[(s-1)^2+1]} = \frac{s}{(s^2+1)(s^2-2s+2)} =$$

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2}$$

$$As^3 - 2As^2 + 2As + Bs^2 - 2Bs + 2B + Cs^3 + Cs + Ds^2 + D = s$$

$A + C = 0$ $\quad s^3$

$-2A + B + D = 0$ $\quad s^2$ $\quad >$

$2A - 2B + C = 1$ $\quad s$

$2B + D = 0$ $\quad 2$

$$\begin{array}{l} -B + C + D = 1 \\ \underline{-2B \quad - D = 0} \\ -3B + C = 1 \end{array}$$

or use a matrix

$C = 1 + 3B$ or $\frac{C-1}{3} = B$

$A = -C$

$$2A - 2B + C = 1 \rightarrow 2(-C) - 2\left(\frac{C-1}{3}\right) + C = 1$$

$$6(-C) - 2(C-1) + 3C = 3$$

$$-6C - 2C + 2 + 3C = 3$$

$$-8C + 3C + 2 = 3$$

$$-5C = 1$$

$$C = -1/5 \quad A = 1/5$$

$$B = \frac{-1/5 - 1}{3} = \frac{-6/5}{3} = -2/5$$

$$2B + D = 0 \rightarrow D = -2B \rightarrow D = -2(-2/5) = 4/5$$

$$Y(s) = \frac{1/5\, s}{s^2+1} - \frac{2/5}{s^2+1} - \frac{1/5\, s}{(s-1)^2+1} + \frac{4/5}{(s-1)^2+1}$$

$$y(t) = \frac{1}{5}\cos t - 2/5 \sin t - 1/5 e^t \cos t + 4/5 e^t \sin t$$