9/19/2022

Euler's Method
Existence and uniqueness
Linear/Integrating factors

Euler's Method is a numerical technique for solving (approximating) first order differential equations using successive linear approximations.

Starting point (initial conditions) (x_0, y_0) or expressed as $y(x_o) = y_0$. And we have a differential equation, $\frac{dy}{dx} = f(x, y)$. Destination, a place we want to approximate the differential equation at, $y(b)$

Additionally, we are typically information about the step size, or the number of steps we want to take to get to the estimation. Step size as $h = \Delta x$. The number steps is given as n.

$$\frac{b-a}{n} = \Delta x = h$$

$$a = x_o$$

Example.
I want to estimate the value of $y(3)$ given that $y' = y(2 - ty), y(2) = 1$ and do it in 3 steps.

$$y_{n+1} = y_n + m_n(\Delta t)$$

$$m_n = y'(t_n, y_n)$$

Step 0:

$$x_0 = 2, y_0 = 1$$
$$m_0 = (1)\big(2 - (2)(1)\big) = 0$$

$$\Delta t = \frac{3-2}{3} = \frac{1}{3}$$

$$y_{n+1} = 1 + 0\left(\frac{1}{3}\right) = 1$$

$$t_{n+1} = t_n + \Delta t$$

Step 1:

$$x_1 = 2 + \frac{1}{3} = \frac{7}{3}, y_1 = 1$$

$$m_1 = (1)\left(2 - \left(\frac{7}{3}\right)(1)\right) = -\frac{1}{3}$$

$$y_2 = 1 + \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{8}{9} \approx 0.888888\ldots$$

$$x_2 = \frac{8}{3}$$

Step 2:

$$x_2 = \frac{8}{3}, y_2 = \frac{8}{9}$$

$$m_2 = \left(\frac{8}{9}\right)\left(2 - \left(\frac{8}{3}\right)\left(\frac{8}{9}\right)\right) = -\frac{80}{243}$$

$$y_3 = \frac{8}{9} + \left(-\frac{80}{243}\right)\left(\frac{1}{3}\right) = \frac{568}{729} \approx 0.779 \ldots$$

$$x_3 = \frac{8}{3} + \frac{1}{3} = 3$$

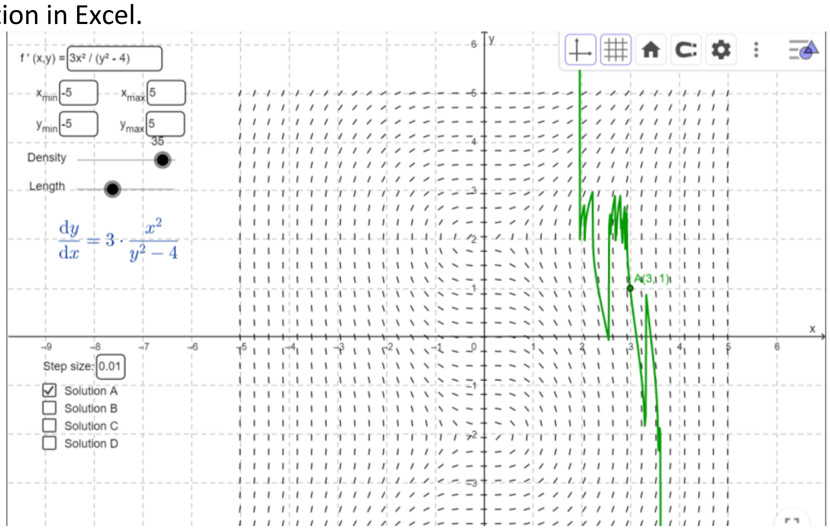
Estimate for $y(3) \approx \frac{568}{729}$ or $0.779 \ldots$

Excel confirms our calculation.

Example.

Estimate the value of $y(7)$ if $\frac{dy}{dt} = \frac{3t^2}{y^2-4}, y(3) = 1$ using a step size of $\Delta t = 0.1$.

Calculation in Excel.



Differential equation has vertical slopes at y=2 and y=-2, can cause unpredictable behavior when using approximation methods.

Existence and Uniqueness

$$\frac{dy}{dx} = f(x, y)$$

Most general form of a differential equation.
Similar to $\frac{dy}{dt} = \frac{3t^2}{y^2-4}$.

If the differential equation is in a different form, you can solve for y'.

When the differential equation is in this form, there is a test for whether or not a solution exists at a given point initial point.

Condition #1: is $f(x, y)$ defined (at the point)
Condition #2: Is $\frac{\partial f}{dy}$ defined?

(i.e. take the derivative of f with y as the variable, and all other variables treated like constants)

Using our example, where is the differential equation defined?
The differential equation will be undefined when the denominator is zero. So the function is not defined when $y = 2, -2$.

Also check the derivative of the function.

$$f(x, y) = \frac{3x^2}{y^2 - 4} = 3x^2(y^2 - 4)^{-1}$$

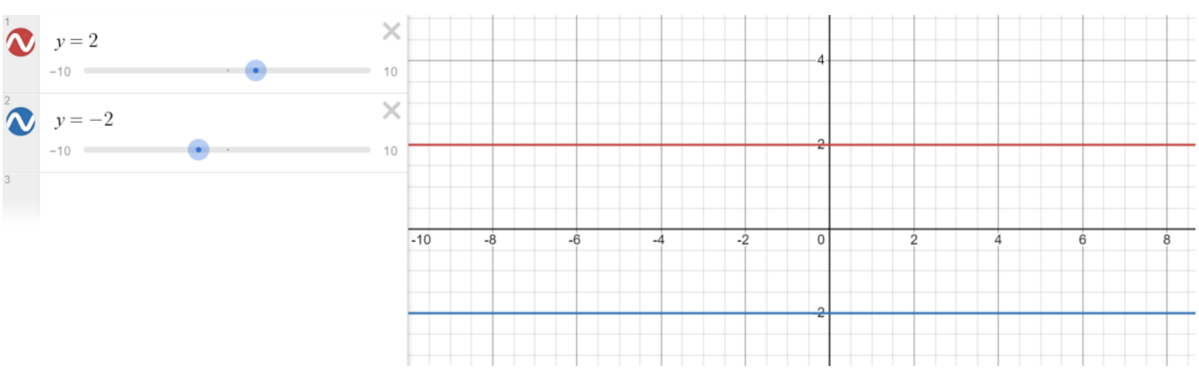
$$f_y(x, y) = 3x^2(-1)(y^2 - 4)^{-2}(2y) = -\frac{6x^2y}{(y^2 - 4)^2}$$

Is this function undefined anywhere? (anywhere that the original function was defined?)

This function is also undefined at $y = 2, -2$ and so does not add any additional restrictions.

A solution (unique solution) exists for this differential anywhere that $y \neq 2$ or $y \neq -2$.

Sometimes you may be asked to plot the regions where the graph is defined.

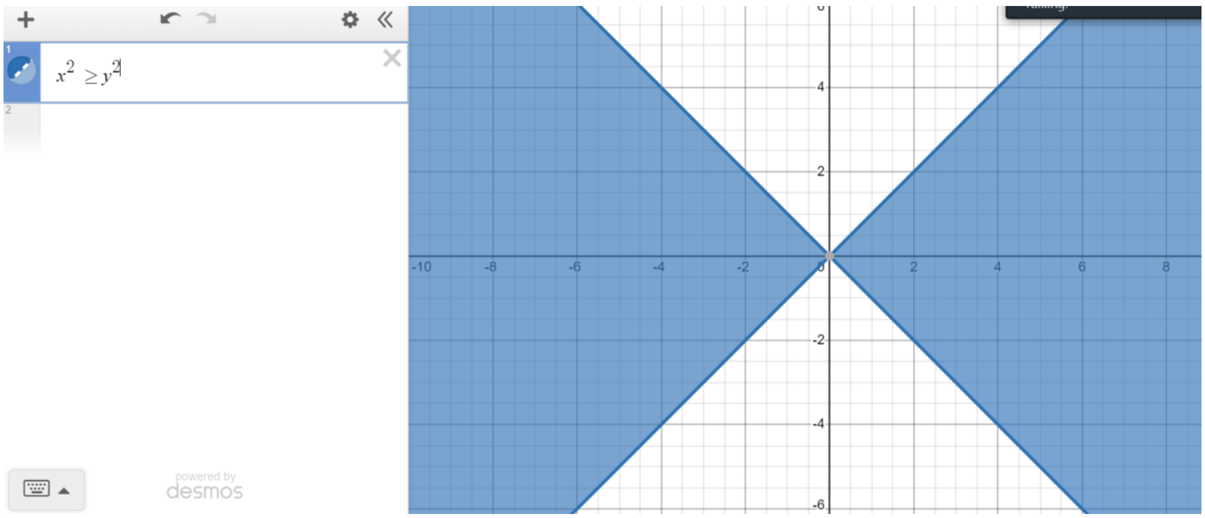


Example. Plot the regions where $\frac{dy}{dx} = \sqrt{x^2 - y^2}$ is defined.

$$f(x, y) = \sqrt{x^2 - y^2}$$

Where is $\sqrt{x^2 - y^2}$ defined?

$$x^2 - y^2 \geq 0$$
$$x^2 \geq y^2$$



$$|x| \geq |y|$$

Defined in shaded regions. Not defined outside shaded regions.

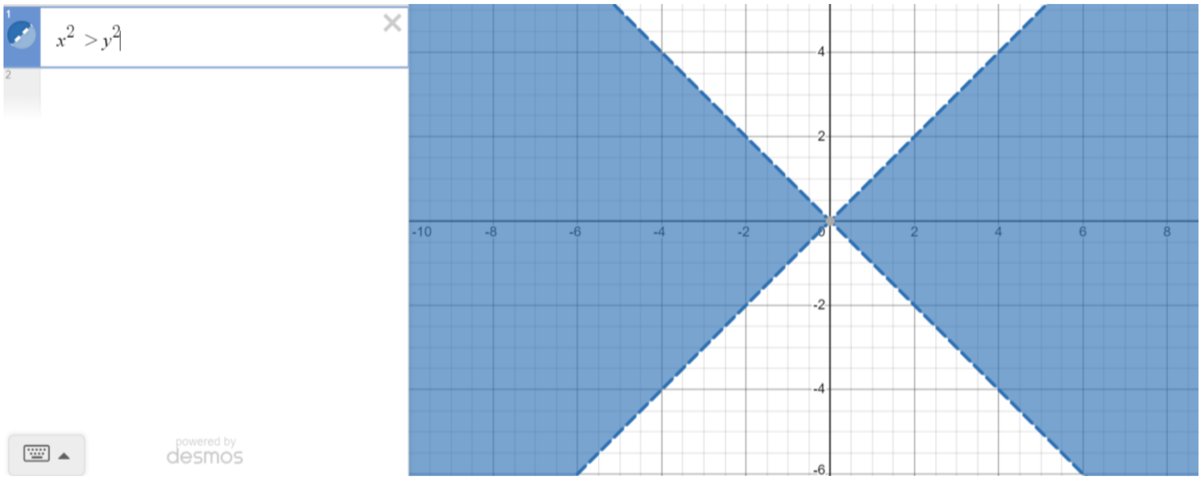
$$f(x, y) = \sqrt{x^2 - y^2} = (x^2 - y^2)^{1/2}$$

$$f_y(x, y) = \frac{1}{2}(x^2 - y^2)^{-\frac{1}{2}}(-2y) = -\frac{y}{\sqrt{x^2 - y^2}}$$

This does add additional restrictions for existence and uniqueness. The original function could permit this square root to be zero, but now that would put a zero in the denominator, and that’s not allowed.

Our new restriction is

$$|x| > |y|$$



The only thing that changed here was the dotted line vs. the solid line.

If you have initial conditions, check to see if the initial points fall into a defined region. Any restrictions could impact how y evolves over time, or a limitation on the advancement of x or t from left to right. If you initial conditions were y(0)=0, and your differential equation had undefined values at t=-2, and t=1, then your solution would only be defined inside the interval (-2,1). If your initial condition was y(3)=0, then you could project forward as far as you wanted, but only backward to t=1. Defined on the interval (1,∞).

In the case of linear differential equations:

$$y'(t) + p(t)y(t) = g(t)$$

Standard form. (if you have anything containing t or x in front of y', divide it out to put the equation in this form).

The existence of a solution will depend on $p(t)$. Where this function is defined is going to determine where the entire differential equation is defined. And the restriction on the solution is only going to depend on t.

$$(1-t)y' + ty = t^2 - 1$$

$$y' + \frac{t}{1-t}y = -t - 1$$

$$p(t) = \frac{t}{1-t}$$

$p(t)$ is undefined when t=1. So, a solution will exist on the interval $(-\infty, 1) \cup (1, \infty)$.
If your initial condition are $y(0) = 0$, then the solution exists on the interval $(-\infty, 1)$. You can only go one step forward in time, but as far back in time as you like. But if your initial conditions are $y(2) = 0$, then your solution is defined on $(1, \infty)$. You can only go back in time to 1, but as far forward in time as you like.

Solutions to Linear Differential Equations.
This method involves an integrating factor: try to find a function that we can multiply through the entire equation in order to make it easy to integrate. The trick involves thinking of one side of our equation as a product rule. Sometimes this is referred to as a “reverse product rule”.

A general linear equation looks like:

$$a(t)y'(t) + b(t)y(t) = f(t)$$

In standard form, $y'(t)$ is by itself, so divide through by $a(t)$

$$y'(t) + \frac{b(t)}{a(t)}y(t) = \frac{f(t)}{a(t)}$$

$$y'(t) + p(t)y(t) = g(t)$$

We want to obtain an integrating that will make the right side of this equation equal to a product rule.

Recall:

$$[(u(t)v(t)]' = u'(t)v(t) + u(t)v'(t)$$

We call the integrating factor $\mu(t) = e^{\int p(t)dt}$

$$\mu'(t) = e^{\int p(t)dt}\left(\int p(t)dt\right)' = e^{\int p(t)dt}p(t)$$

Multiply through by $\mu(t)$

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = g(t)\mu(t)$$
$$\mu(t)y'(t) + \mu'(t)y(t) = g(t)\mu(t)$$

$$\big(\mu(t)y(t)\big)' = g(t)\mu(t)$$

$$\int \big(\mu(t)y(t)\big)' dt = \int g(t)\mu(t)dt$$

$$\mu(t)y(t) = \int g(t)\mu(t)dt + C$$

$$y(t) = \frac{1}{\mu(t)}\left[\int g(t)\mu(t)dt + C\right]$$

Variation of parameters formula.

Example.
Solve $ty' + 2y = t^2 - t + 1, y(1) = \frac{1}{2}$

$$ty' + 2y = t^2 - t + 1$$
$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2\left(y' + \frac{2}{t}y\right) = t^2\left(t - 1 + \frac{1}{t}\right)$$

$$t^2 y' + 2ty = t^3 - t^2 + t$$
$$(t^2 y)' = t^3 - t^2 + t$$

$$\int (t^2 y)' dt = \int t^3 - t^2 + t dt$$

$$t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$$

$$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}$$
$$y(1) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4}(1)^2 - \frac{1}{3}(1) + \frac{1}{2} + \frac{C}{(1)^2}$$
$$C = \frac{1}{12}$$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

$$y(t) = \frac{1}{\mu(t)}\left[\int g(t)\mu(t) dt + C\right]$$

$$\mu(t) = t^2, g(t) = t - 1 + \frac{1}{t}$$

$$y(t) = \frac{1}{t^2}\left[\int \left(t - 1 + \frac{1}{t}\right) t^2 dt + C\right] = \frac{1}{t^2}\left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C\right] = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}$$

Example.
Solve $2y' - y = 4\sin(3t)\,, y(0) = 2$

$$y' - \frac{1}{2}y = 2\sin(3t)$$

$$\mu(t) = e^{\int -\frac{1}{2} dt} = e^{-\frac{1}{2}t}$$

$$e^{-\frac{1}{2}t} y' - \frac{1}{2} e^{-\frac{1}{2}t} y = 2e^{-\frac{1}{2}t}\sin(3t)$$

$$\left(e^{-\frac{1}{2t}}y\right)' = 2e^{-\frac{1}{2}t}\sin(3t)$$

$$\int \left(e^{-\frac{1}{2t}}y\right)' dt = \int 2e^{-\frac{1}{2}t}\sin(3t)\,dt$$

$$e^{-\frac{1}{2}t}y = \int 2e^{-\frac{1}{2}t}\sin(3t)\,dt$$

$$\int 2e^{-\frac{1}{2}t}\sin(3t)\,dt$$

$$u = 2\sin(3t)\,,dv = e^{-\frac{1}{2}t}dt$$
$$du = 6\cos(3t)\,dt, v = -2e^{-\frac{1}{2}t}$$

$$2\sin(3t)\left[-2e^{-\frac{1}{2}t}\right] - \int -2e^{-\frac{1}{2}t}(6\cos(3t))dt$$

$$-4e^{-\frac{1}{2}t}\sin(3t) + \int 12e^{-\frac{1}{2}t}\cos(3t)\,dt$$

$$u = 12\cos(3t)\,,dv = e^{-\frac{1}{2}t}dt$$
$$du = -36\sin(3t)\,dt, v = -2e^{-\frac{1}{2}t}$$

$$-4e^{-\frac{1}{2}t}\sin(3t) + 12\cos(3t)\left[-2e^{-\frac{1}{2}t}\right] - \int -2e^{-\frac{1}{2}t}(-36\sin(3t))dt$$

$$\int 2e^{-\frac{1}{2}t}\sin(3t)\,dt = -4e^{-\frac{1}{2}t}\sin(3t) + 12\cos(3t)\left[-2e^{-\frac{1}{2}t}\right] - \int 72e^{-\frac{1}{2}t}\sin(3t)\,dt$$

$$\int 2e^{-\frac{1}{2}t}\sin(3t)\,dt + \int 72e^{-\frac{1}{2}t}\sin(3t)\,dt = -4e^{-\frac{1}{2}t}\sin(3t) + 12\cos(3t)\left[-2e^{-\frac{1}{2}t}\right]$$

$$37\int 2e^{-\frac{1}{2}t}\sin(3t)\,dt = -4e^{-\frac{1}{2}t}\sin(3t) - 24\cos(3t)\left[e^{-\frac{1}{2}t}\right]$$

$$\int 2e^{-\frac{1}{2}t}\sin(3t)\,dt = \frac{1}{37}\left[-4e^{-\frac{1}{2}t}\sin(3t) - 24e^{-\frac{1}{2}t}\cos(3t)\right] + C$$

$$e^{-\frac{1}{2}t}y = \int 2e^{-\frac{1}{2}t}\sin(3t)\,dt$$

$$e^{-\frac{1}{2}t}y = \frac{1}{37}\left[-4e^{-\frac{1}{2}t}\sin(3t) - 24e^{-\frac{1}{2}t}\cos(3t)\right] + C$$

Multiply everything by $e^{\frac{1}{2}t}$

$$y = -\frac{4}{37}\sin(3t) - \frac{24}{37}\cos(3t) + Ce^{\frac{1}{2}t}$$

$$2 = -\frac{4}{37}(0) - \frac{24}{37}(1) + Ce^{0}$$

$$C = 2 + \frac{24}{37} = \frac{98}{37}$$

$$y = -\frac{4}{37}\sin(3t) - \frac{24}{37}\cos(3t) + \frac{98}{37}e^{\frac{1}{2}t}$$

Tank Problems/Concentration Problems

We have a large tank, that is filled either with water (some other fluid) or a mixture of water and some other chemical that dissolves into the water. (well-mixed mixture). Add additional fluid or mixture to the tank at a particular rate, and then removed the mixture (after mixing) from the tank (at a particular rate).

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

$$Rate_{in} = \frac{quantity\ of\ mixed\ product}{unit\ of\ liquid} \times \frac{unit\ of\ liquid}{per\ unit\ of\ time}$$

$\frac{quantity\ of\ mixed\ product}{unit\ of\ liquid}$ think 5 g of salt per liter.

$\frac{unit\ of\ liquid}{per\ unit\ of\ time}$ think 3 liters per minute

Rate out is similar, but we have to take into the size of the tank, and that the amount (of salt) may be increasing or decreasing.
The second product is basically the same as for rate in.

Basically, there are two scenarios that typically happen in a problem like this:
Rate of liquid flowing in and flowing out is identical, in which case, the problem can be solved with separation of variables (there is no time dependency in the rate out term).

If the rates of flow in and out are different, then the amount of liquid in the tank will depend on time, and we'll need to use linear solution methods to solve it.
(We tend to get powers of the denominator of the rate out term), and coefficients that are very large.