11/7/2022

Circuit Problems
Spring Problems

Circuit Problems: a typical RLC circuit, in terms of charge.

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

We will look at homogeneous case where $E(t)$ is 0.

L = inductance, henris
R = resistance, ohms
C = capacitance, farads
q = charge
E = energy (forcing function) in volts

Example. Find the charge on an RLC circuit where the inductance is 5/3 H, the resistance is 100 ohms, and capacitance is 1/30 F. Initial charge is 0 ($q(0) = 0$), and the initial current $q'(0)$=9 amps.

$$\frac{5}{3}q'' + 100q' + 30q = 0$$

$$5q'' + 300q' + 90q = 0$$
$$q'' + 60q' + 18q = 0$$

$$k^2 + 60k + 18 = 0$$

$$k = \frac{-60 \pm \sqrt{60^2 - 4(1)(18)}}{2} = \frac{-60 \pm \sqrt{3528}}{2} = \frac{-60 \pm 2\sqrt{882}}{2} = -30 \pm \sqrt{882} =$$
$$-30 \pm 3\sqrt{98} = -30 \pm 21\sqrt{2}$$

Homogeneous solution

$$q(t) = c_1 e^{(-30+21\sqrt{2})t} + c_2 e^{(-30-21\sqrt{2})t}$$
$$q(0) = 0 \to 0 = c_1 + c_2$$

$$q'(t) = \left(-30 + 21\sqrt{2}\right)c_1 e^{(-30+21\sqrt{2})t} + \left(-30 - 21\sqrt{2}\right)c_2 e^{(-30-21\sqrt{2})t}$$

$$q'(0) = 9 \to 9 = \left(-30 + 21\sqrt{2}\right)c_1 + \left(-30 - 21\sqrt{2}\right)c_2$$

Since

$$c_1 + c_2 = 0 \to c_1 = -c_2$$

$$\left(-30 + 21\sqrt{2}\right)(-c_2) + \left(-30 - 21\sqrt{2}\right)c_2 = 9$$

$$30c_2 - 21\sqrt{2}c_2 - 30c_2 - 21\sqrt{2}c_2 = 9$$
$$-42\sqrt{2}c_2 = 9$$

$$c_2 = -\frac{9}{42\sqrt{2}} = -\frac{3}{14\sqrt{2}} = -\frac{3\sqrt{2}}{28}$$

$$c_1 = \frac{3\sqrt{2}}{28}$$

$$q(t) = \frac{3\sqrt{2}}{28}e^{(-30+21\sqrt{2})t} - \frac{3\sqrt{2}}{28}e^{(-30-21\sqrt{2})t}$$

Springs.

Form of the spring equation:

$$my'' + \gamma y' + ky = F(t)$$

y = position, in SI is meters, is feet in imperial/English
m = mass, kilograms in SI, and in slugs in imperial/English
γ = damping coefficient (often given in terms of scale of velocity)
k = spring constant
F = forcing function, newtons in SI, or pounds in imperial/English

Example.
Assume an object weighing 2 lb stretches a spring 6 in. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 16 ft/sec. What is the period of the motion?

When there is no damping, this is called a simple harmonic oscillator; an undamped spring. The solution is imaginary (cosines and sines only).

First find the spring constant

$$F = ky$$
$$2 = k\left(\frac{1}{2}\right)$$

6 in = ½ foot

$$k = 4$$

$$F = ma$$
$$a = 32\frac{ft}{s^2}$$

$$\frac{2}{32} = m = \frac{1}{16} slugs$$

No damping so gamma is zero.

$$\frac{1}{16}y'' + 4y = 0$$

$$y'' + 64y = 0$$
$$k^2 + 64 = 0$$

$$k = \pm 8i$$

$$y(t) = c_1 \cos(8t) + c_2 \sin(8t)$$

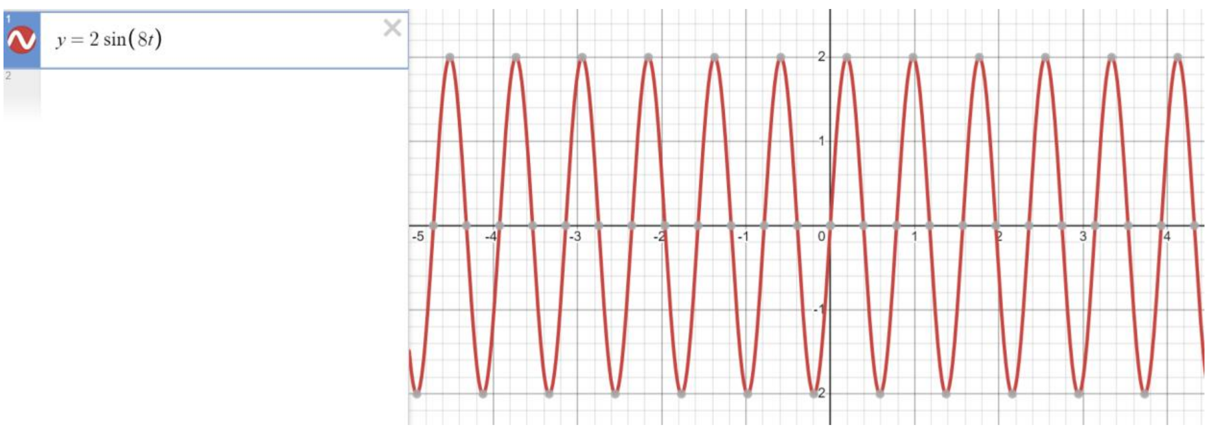
Equilibrium $y(0) = 0$

$$y'(0) = 16$$

$c_1 = 0$ because cos(0)=1

$$y'(t) = 8c_2 \cos(8t)$$
$$16 = 8c_2(1) \rightarrow c_2 = 2$$

$$y(t) = 2\sin(8t)$$



Here, Amplitude=2

$$y(t) = Acos(\omega t) + Bsin(\omega t)$$

$$Amp = \sqrt{A^2 + B^2}$$

ω is the frequency. Period $T = \frac{2\pi}{\omega}$

Undamped problems have homogeneous solutions that contain only trig functions (no exponentials). Therefore, they don't contain transient solutions. Steady state solutions will persist. In damped problems, the steady state solutions will come from the forcing term, but not in the undamped case.

When there is a forcing term, we can get two special phenomena to appear in the undamped case.

Resonance:
When the forcing function and the homogeneous have the same natural frequency. (the frequency in a damped problem is called the quasi-frequency).

Recall that if our solutions to the homogeneous equation is $y_1 = \cos(8t)$, $y_2 = \sin(8t)$ and your forcing function is $F(t) = \sin(8t)$, then when we guess the form of the non-homogeneous solution we multiply by t. That t increases the amplitude over time.

$$\frac{1}{16}y'' + 4y = \sin(8t)$$

$$y'' + 64y = 16\sin(8t)$$

$$y(t) = c_1\cos(8t) + c_2\sin(8t)$$

$$y_p(t) = At\sin(8t) + Bt\cos(8t)$$

$$y_p' = A\sin(8t) + 8At\cos(8t) + B\cos(8t) - 8Bt\sin(8t)$$

$$y_p'' = 8A\cos(8t) + 8A\cos(8t) - 64At\sin(8t) - 8B\sin(8t) - 8B\sin(8t) - 64Bt\cos(8t)$$

$$= 16A\cos(8t) - 64At\sin(8t) - 16B\sin(8t) - 64Bt\cos(8t)$$

$$16A\cos(8t) - 64At\sin(8t) - 16B\sin(8t) - 64Bt\cos(8t) + 64(At\sin(8t) + Bt\cos(8t)) = 16\sin(8t)$$

$$16A\cos(8t) - 16B\sin(8t) = 16\sin(8t)$$

$$A = 0, B = -1$$

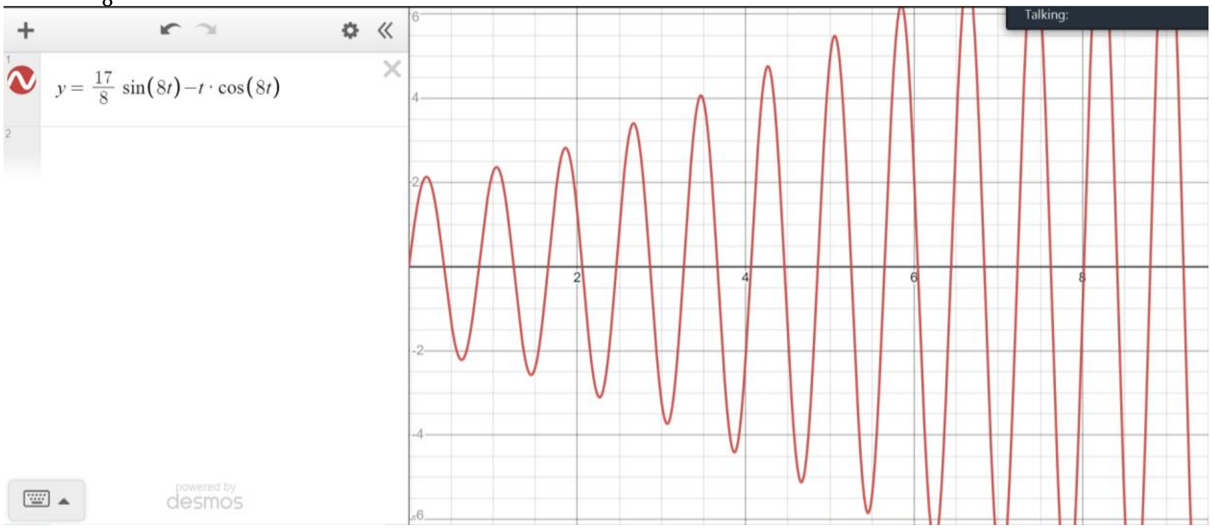
$$y(t) = c_1\cos(8t) + c_2\sin(8t) - t\cos(8t)$$

$$0 = c_1 + 0 + 0$$

$$y(t) = c_2\sin(8t) - t\cos(8t)$$
$$y'(t) = 8c_2\cos(8t) - \cos(8t) + 8t\sin(8t)$$

$$16 = 8c_2 - 1$$
$$17 = 8c_2$$
$$c_2 = \frac{17}{8}$$

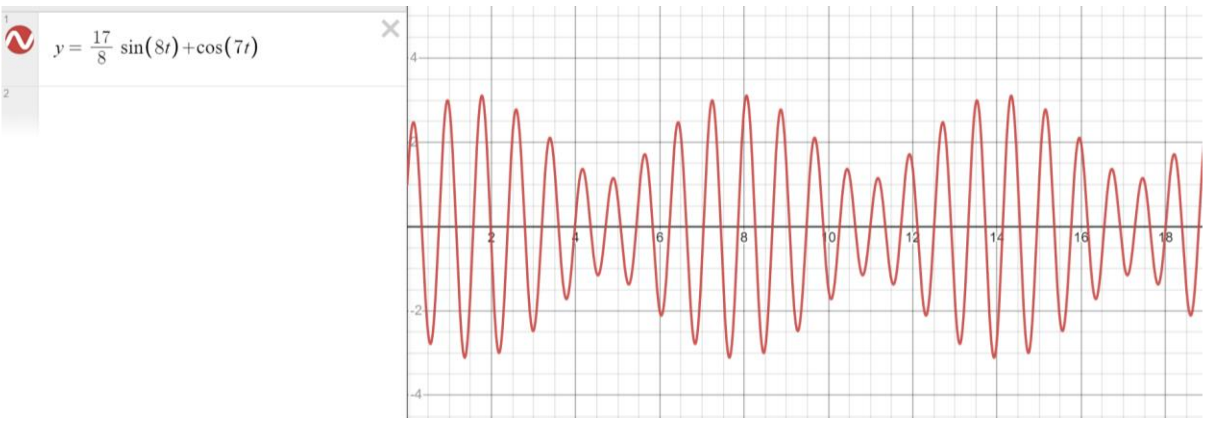
$y(t) = \frac{17}{8}\sin(8t) - t\cos(8t)$



Beats

When the natural frequency and the forcing frequency are similar but not identical.

Suppose the solution to the system was like $y(t) = \sin(8t) + \cos(7t)$



Damped systems:

Underdamped systems (there is a little damping, but there is still a decreasing oscillation)

Critically damped system (there are repeated roots in the solution)

Overdamped system (there is a lot of damping, and real roots)

Critically damped and overdamped systems can only cross 0 one time.

Example.

A 2-kg mass is attached to a spring with spring constant 24 N/m. The system is then immersed in a medium imparting a damping force equal to 16 times the instantaneous velocity of the mass. Find the equation of motion if it is released from rest at a point 40 cm below equilibrium.

$$2y'' + 16y' + 24y = 0$$
$$y'(0) = 0, y(0) = -0.4$$

$$y'' + 8y' + 12y = 0$$
$$k^2 + 8k + 12 = 0$$

$$(k+6)(k+2) = 0$$
$$k = -2, -6$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-6t}$$

Overdamped

$$c_1 + c_2 = -0.4$$

$$y'(t) = -2c_1 e^{-2t} - 6c_2 e^{-6t}$$

$$0 = -2c_1 - 6c_2$$

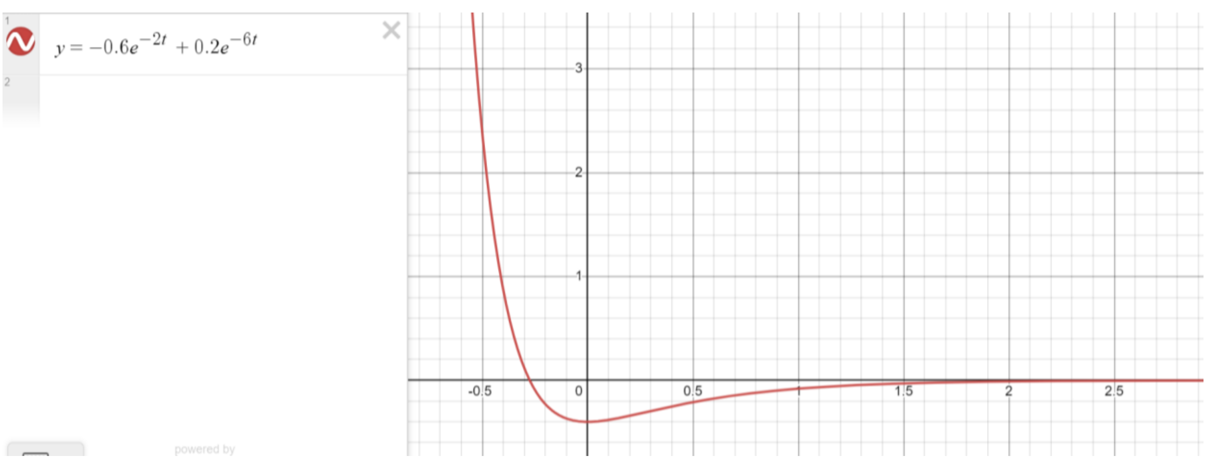
$$2c_1 + 2c_2 = -0.8$$
$$-2c_2 - 6c_2 = 0$$

$$-4c_2 = -0.8$$

$$c_2 = 0.2$$

$$c_1 + 0.2 = -0.4$$
$$c_1 = -0.6$$

$$y(t) = -0.6e^{-2t} + 0.2e^{-6t}$$



A critically damped system

$$y(t) = c_1 e^{-rt} + c_2 t e^{-rt}$$

Looks somewhat similar in a graph.

Decrease the damping.

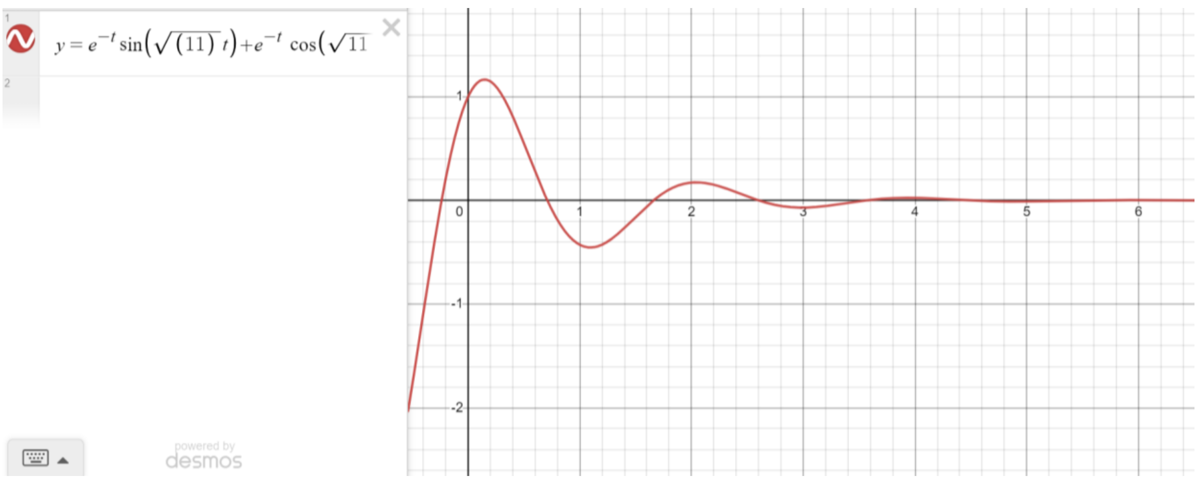
Example.
A 2-kg mass is attached to a spring with spring constant 24 N/m. The system is then immersed in a medium imparting a damping force equal to 4 times the instantaneous velocity of the mass. Find the equation of motion if it is released from rest at a point 40 cm below equilibrium.

$$2y'' + 4y' + 24y = 0$$
$$y'(0) = 0, y(0) = -0.4$$

$$y'' + 2y' + 12y = 0$$
$$k^2 + 2k + 12 = 0$$

$$k = \frac{-2 \pm \sqrt{2^2 - 4(1)(12)}}{2(1)} = \frac{-2 \pm \sqrt{-44}}{2} = \frac{-2 \pm 2\sqrt{11}i}{2} = -1 \pm \sqrt{11}i$$

$$y(t) = c_1 e^{-t}\cos(\sqrt{11}t) + c_2 e^{-t}\sin(\sqrt{11}t)$$



The quasi-frequency is $\sqrt{11}$ – the period is longer because of the damping.

All damped solutions are transient and eventually (practical standpoint) disappear.

Things I could ask:
Set up/solve a problem, with or without forcing term – in proper second order differential equation form
Resonance vs. beats: when do they occur, why do they matter, etc.
Transient vs. steady state solutions
Damping and behavior of the system: does it oscillate/how many times it cross zero; graph the solution, maximum amplitude

Exam #2:
Second order problems:
- Constant coefficient – assumption e^{kt}
- Cauchy-Euler problems – assumption t^n

Real solutions (distinct)
Repeated solutions
Complex Solutions.

Remember to look up the quadratic formula

Non-homogeneous solutions:

Undetermined coefficients (constant coefficient problem; forcing functions has to contain only polynomials, exponential or sine/cosine)

Variation of parameters method

Wronskian, fundamental sets/independence of solutions & Abel’s theorem
Reduction of order
Higher order problems – solving one problem (homogeneous probably) with third order or higher, factorable.
Circuit problem similar to the example from today
Spring problem: one solving/some identification of damping/beats/resonance