11/21/2022

Laplace Transforms Definition (8.1)
Using Laplace Transform Tables
Inverse Laplace Transforms (8.2)

The definition of the Laplace Transform.

$$L(f) = F(s) = \int_0^\infty e^{-st} f(t) dt$$

The transformation produces a function of s instead of t. This will allow us to solve differential equations using mostly algebra rather than integration or differentiation.

We need to be able to use the definition when called for.

Example.
Find the Laplace Transform of $f(t) = t$.

$$L(t) = F(s) = \int_0^\infty te^{-st} dt =$$

$$u = t, dv = e^{-st} dt$$
$$du = dt, v = -\frac{1}{s} e^{-st}$$

$$\int_0^\infty te^{-st} dt = -\frac{1}{s} te^{-st} - \int_0^\infty -\frac{1}{s} e^{-st} dt = -\frac{1}{s} te^{-st} + \left(-\frac{1}{s} \times \frac{1}{s} e^{-st}\right)\Big|_0^\infty$$

$$-\frac{1}{s} te^{-st} + \left(-\frac{1}{s^2} e^{-st}\right)\Big|_0^\infty = 0 + 0 + 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

$$\lim_{t\to\infty} te^{-st} = \lim_{t\to\infty} \frac{t}{e^{st}} = \lim_{t\to\infty} \frac{1}{se^{st}} = 0$$

$$L(t) = F(s) = \frac{1}{s^2}$$

Example.
Find the Laplace Transform of $f(t) = \sin t$ using the definition.

$$L(\sin t) = F(s) = \int_0^\infty e^{-st} \sin t \, dt =$$

$$u = \sin t \, , dv = e^{-st} dt$$
$$du = \cos t \, dt, v = -\frac{1}{s} e^{-st}$$

$$\int_0^\infty e^{-st}\sin t\,dt = -\frac{1}{s}e^{-st}\sin(t) - \int_0^\infty -\frac{1}{s}e^{-st}\cos t\,dt == -\frac{1}{s}e^{-st}\sin(t) + \frac{1}{s}\int_0^\infty e^{-st}\cos t\,dt$$

$$u = \cos t\,, dv = e^{-st}dt$$
$$du = -\sin t\,dt, v = -\frac{1}{s}e^{-st}$$

$$-\frac{1}{s}e^{-st}\sin(t) + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\cos t - \int_0^\infty \frac{1}{s}e^{-st}\sin t\,dt\right]$$

$$\int_0^\infty e^{-st}\sin t\,dt = -\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t - \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt$$

Add $\frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt$ to both sides.

$$\int_0^\infty e^{-st}\sin t\,dt + \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt$$
$$= -\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t - \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt + \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt$$

$$\left(1 + \frac{1}{s^2}\right)\int_0^\infty e^{-st}\sin t\,dt = -\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\Big|_0^\infty$$

$$\left(\frac{s^2+1}{s^2}\right)\int_0^\infty e^{-st}\sin t\,dt = 0 + 0 + 0 + \frac{1}{s^2}$$

$$\int_0^\infty e^{-st}\sin t\,dt = \left(\frac{1}{s^2}\right)\times\left(\frac{s^2}{s^2+1}\right) = \frac{1}{s^2+1}$$

Example.
Find the Laplace transform of $f(t) = e^{-2t}$ using the definition.

$$L(e^{-2t}) = F(s) = \int_0^\infty e^{-st}e^{-2t}dt = \int_0^\infty e^{-st-2t}dt = \int_0^\infty e^{-(s+2)t}dt = e^{-(s+2)t}\left(-\frac{1}{s+2}\right)\Big|_0^\infty$$

$$= 0 + \left(\frac{1}{s+2}\right) = \frac{1}{s+2}$$

Normally, we don’t use the definition to apply the transformation to a specific function. Instead, use a table of common Laplace transforms to do the conversion.
(Functions of t are in the “time” domain, and functions of s are in the “frequency” domain.)

How do we use the Table:

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |

See last example.
Find the Laplace Transform of $f(t) = e^{-2t}$. $a = -2$
Based on the table, the transform $F(s) = \frac{1}{s-(-2)} = \frac{1}{s+2}$
This matches what we found from the integration.

What about the one before that? The Laplace Transform of $f(t) = \sin t$?

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ |

$a = 1$ for the transformation.

$$F(s) = \frac{1}{s^2 + (1)^2} = \frac{1}{s^2+1}$$

That is what is what we found from the second example.

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |

What is the Laplace Transform of $f(t) = t = t^1$, so n=1.

$$F(s) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

This is exactly what we found when we did the integration the long way in the first example.

$$n! = n(n-1)(n-2)\ldots.(3)(2)(1)$$

$$7! = 7(6)(5)(4)(3)(2)(1) = 5040$$

$$0! = 1$$

Find the Laplace Transform of $f(t) = 6t^3$?

Now n=3.

$$F(s) = 6\left[\frac{3!}{s^{3+1}}\right] = 6\left(\frac{6}{s^4}\right) = \frac{36}{s^4}$$

Example.
What is the Laplace Transform of $f(t) = e^{3t}\sin 2t + te^t$

| | |
|---|---|
| $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $t^n e^{at}$<br>$n = 1,2,3,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ |

$$f_1(t) = e^{3t}\sin 2t$$
$$a = 3, b = 2$$

$$F_1(s) = \frac{2}{(s-3)^2+2^2} = \frac{2}{(s-3)^2+4}$$

$$f_2(t) = te^t$$
$$n = 1, a = 1$$

$$F_2(s) = \frac{1!}{(s-1)^{1+1}} = \frac{1}{(s-1)^2}$$

$$F(s) = \frac{2}{(s-3)^2+4} + \frac{1}{(s-1)^2}$$

Find the Laplace transform of $f(t) = te^t$ using the formula

$$tf(t) \to -\frac{dF(s)}{ds}$$

What is the Laplace transform of $e^t$? We get $F(s) = \frac{1}{s-1}$

$$\frac{dF(s)}{ds} = \frac{d}{ds}[(s-1)^{-1}] = -1(s-1)^{-2} = -\frac{1}{(s-1)^2}$$

$$te^t \to -\left(-\frac{1}{(s-1)^2}\right) = \frac{1}{(s-1)^2}$$

Inverse Laplace Transforms.
We have a function in the s-domain, and we want to convert it back to the t domain.

$$L^{-1}(F) = f(t)$$

Example.
Find the inverse Laplace Transform of the function $F(s) = \frac{1}{s-5}$.

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ |

Identify a=5.

$$f(t) = e^{5t}$$

Example.

$$F(s) = \frac{3s}{s^2+4} = 3\left(\frac{s}{s^2+4}\right) = 3\left(\frac{s}{s^2+2^2}\right)$$

What is $f(t)$?

| | |
|---|---|
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ |

a=2

$$f(t) = 3\cos(2t)$$

Example.

Find the inverse Laplace Transform of $F(s) = \frac{1}{s^3} = \frac{1}{s^{2+1}} = \frac{1}{2}\left(\frac{2}{s^{2+1}}\right)$

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |

n=2
2! =2

$$f(t) = \frac{1}{2}t^2$$

Solving differential equations using Laplace Transforms.

Using Laplace transforms will require using initial conditions, and very specifically, they have to be at t=0.

Example.
Use Laplace transforms to solve

$$y'' - 6y' + 5y = 3e^{2t}, y(0) = 2, y'(0) = 3$$

| $\boldsymbol{f'(t)}$ | $sF(s)-f(0)$ |
|---|---|
| $\boldsymbol{f''(t)}$ | $s^2F(s)-sf(0)$<br>$-f'(0)$ |

$$y'' = s^2Y(s) - s(2) - 3$$

$$y' = sY(s) - 2$$

$$y = Y(s)$$

$$e^{2t} = \frac{1}{s-2}$$

$$s^2Y(s) - s(2) - 3 - 6(sY(s) - 2) + 5Y(s) = \frac{3}{s-2}$$

$$s^2Y(s) - 2s - 3 - 6sY(s) + 12 + 5Y(s) = \frac{3}{s-2}$$

$$s^2Y(s) - 2s - 6sY(s) + 9 + 5Y(s) = \frac{3}{s-2}$$

$$s^2Y(s) - 6sY(s) + 5Y(s) = \frac{3}{s-2} + \frac{(2s-9)(s-2)}{s-2}$$

$$Y(s)(s^2 - 6s + 5) = \frac{3 + 2s^2 - 4s - 9s + 18}{s-2}$$

$$Y(s) = \frac{2s^2 - 13s + 21}{(s-2)(s^2 - 6s + 5)} = \frac{2s^2 - 13s + 21}{(s-2)(s-5)(s-1)}$$

$$\frac{2s^2 - 13s + 21}{(s-2)(s-5)(s-1)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{C}{s-1}$$

$$2s^2 - 13s + 21 = A(s-5)(s-1) + B(s-2)(s-1) + C(s-2)(s-5)$$

$$= A(s^2 - 6s + 5) + B(s^2 - 3s + 2) + C(s^2 - 7s + 10)$$

$$= As^2 - 6As + 5A + Bs^2 - 3Bs + 2B + Cs^2 - 7Cs + 10C = 2s^2 - 13s + 21$$

$$A + B + C = 2 \ \ (s^2)$$
$$-6A - 3B - 7C = -13 \ \ (s)$$
$$5A + 2B + 10C = 21 \ \ (1)$$

Use systems of equations (substitution or elimination) or matrix methods to solve for A, B, C.

$$A = -1, B = \frac{1}{2}, C = \frac{5}{2}$$

$$Y(s) = \frac{-1}{s-2} + \frac{\left(\frac{1}{2}\right)}{s-5} + \frac{\left(\frac{5}{2}\right)}{s-1}$$

Finally, apply the inverse Laplace transform to get back to functions of t.

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ |

$$y(t) = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

Old way:

1) Find the homogeneous solution
2) Find the particular solution for the forcing function
3) Apply the initial conditions.

Laplace Transform way:

1) Transform to the s-domain.
2) Transform back to the t-domain.

Next week, unit step functions and convolutions.