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Non-homogeneous Solutions

Undetermined Coefficients
Variation of Parameters

Homogeneous Second Order Differential equations

$$a(x)y'' + b(x)y' + c(x)y = 0$$

Constant Coefficients case where $a(x) = a, b(x) = b, c(x) = c$.
Cauchy-Euler case, where $a(x) = ax^2, b(x) = bx, c(x) = c$.

Non-homogeneous Second Order Differential Equation

$$a(x)y'' + b(x)y' + c(x)y = F(x)$$

Variation of Parameters can apply to any type of 2nd order (or higher) type of equation, with any forcing function, and constant coefficient, Cauchy-Euler or any other situation for which we have a fundamental set of solutions.

Undetermined Coefficients:
Only works for functions that are constant coefficient differential equations.

$$ay'' + by' + cy = F(x)$$

This method only applies to certain kinds of forcing functions, $F(x)$.

The general kind of function that works with undetermined coefficients has the property that the set of derivatives of the function is finite (give or take a constant).

- Polynomials: $x^n \to nx^{n-1} \to n(n-1)x^{n-2} \to \cdots \to n!\,x^0 \to 0$ (as long as n is a positive integer, and $n! = n(n-1)(n-2)\ldots(3)(2)(1)$)
- Exponential functions: $e^{ax} \to ae^{ax} \to a^2e^{ax}$ … (all derivatives are constant multiples of the original e^{ax})
- Some Trig functions (sine and cosine): $\sin(x) \to \cos(x) \to -\sin(x) \to -\cos(\mathrm{x}) \to \sin(\mathrm{x}) \to \cdots$(sine and cosine are the only two functions in this list, all others are constant multiples)
- Products of any of these functions: $x^2e^{2x}, e^x\sin(x), etc.$

General Strategy for the method of underdetermine coefficients:

1) Solve the homogeneous differential equation first (there can be an interaction between the homogenous solution and the non-homogeneous solution—so we need to know the homogeneous solution first).
2) Guess a function, called Ansatz, $Y(x)$, $y_p(x)$, based on the form of the forcing function, with unknown coefficients
3) Plug in the Ansatz into the differential equation, and solve for the unknown constants.
4) Solution to the non-homogeneous ODE is the homogeneous solution + the non-homogeneous component.

Picking the Ansatz

If $F(x)$ contains	$Y(x)$ should be
A polynomial (highest degree) x^3	Starting with the highest degree, and all lower degrees to the constant: $Ax^3 + Bx^2 + Cx + D$
Exponential e^{ax}	Just the exponential function Ae^{ax} Match the exponent
Either a sine, cosine or both $\sin(dx) + \cos(dx)$	Include both sine and cosine $Asin(dx) + Bcos(dx)$ Match the frequency
A product, then for each component of the product x^2e^{2x} $e^x \sin(3x)$	make the usual guess (from above) and multiply $(Ax^2 + Bx + C)e^{2x} = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}$ $Ae^x \sin(3x) + Be^x \cos(3x)$
If the homogenous solution matches the forcing function (in form) $y_1 = e^{2x}, F(x) = e^{2x}$	Multiple by x to obtain the Ansatz $Y(x) = Axe^{2x}$

Example.

$$y'' - 4y' - 12y = 3e^{5t}$$

1) Solve the homogeneous equation.

$$y'' - 4y' - 12y = 0$$
$$k^2 - 4k - 12 = 0$$
$$(k-6)(k+2) = 0$$

$$k = 6, -2$$
$$y_h(t) = c_1e^{6t} + c_2e^{-2t}$$

2) Guess the Ansatz

$$F(t) = 3e^{5t}$$
$$Y(t) = Ae^{5t}$$

Check that this doesn't match one of the homogeneous solutions. If it does not, then proceed. If it does, multiply x until you get an independent function.

3) Plug into the ODE and solve for the constants,

$$Y(t) = Ae^{5t}, Y'(t) = 5Ae^{5t}, Y''(t) = 25Ae^{5t}$$

$$y'' - 4y' - 12y = 3e^{5t}$$

$$25Ae^{5t} - 4(5Ae^{5t}) - 12(Ae^{5t}) = 3e^{5t}$$

$$e^{5t}(25A - 20A - 12A) = 3e^{5t}$$

$$(-7A) = 3$$
$$A = -\frac{3}{7}$$

$$Y(t) = -\frac{3}{7}e^{5t}$$

Full solution (before solving for any initial values)

$$y(t) = y_h(t) + Y(t)$$
$$y(t) = c_1 e^{6t} + c_2 e^{-2t} - \frac{3}{7}e^{5t}$$

If you need to solve for c_1 and c_2, you need to do that after solving for $Y(t)$.

Example.

$$y'' - 4y' - 12y = \sin(2t)$$

1) Solve the homogeneous equation.

$$y'' - 4y' - 12y = 0$$
$$k^2 - 4k - 12 = 0$$
$$(k-6)(k+2) = 0$$

$$k = 6, -2$$
$$y_h(t) = c_1 e^{6t} + c_2 e^{-2t}$$

2) Guess Ansatz

$$y'' - 4y' - 12y = \sin(2t)$$

$$Y(t) = A\sin(2t) + B\cos(2t)$$

3) Plug into the ODE

$$Y(t) = A\sin(2t) + B\cos(2t)\,, Y'(t) = 2A\cos(2t) - 2B\sin(2t)\,, Y''(t) = -4A\sin(2t) - 4B\cos(2t)$$

$$y'' - 4y' - 12y = \sin(2t)$$

$$-4A\sin(2t) - 4B\cos(2t) - 4(2A\cos(2t) - 2B\sin(2t)) - 12(A\sin(2t) + B\cos(2t)) = \sin(2t)$$

$$[-4Asin(2t) + 8B\sin(2t) - 12A\sin(2t)] + [-4B\cos(2t) - 8A\cos(2t) - 12B\cos(2t)] = \sin(2t)$$

$$(-4A + 8B - 12A)\sin(2t) + (-4B - 8A - 12B)\cos(2t) = (1)\sin(2t) + (0)\cos(2t)$$

$$-16A + 8B = 1$$
$$-8A - 16B = 0$$

$$-8A = 16B$$
$$A = -2B$$

$$-16(-2B) + 8B = 1$$
$$32B + 8B = 1$$
$$40B = 1$$
$$B = \frac{1}{40}$$
$$A = -\frac{1}{20}$$

$$Y(t) = -\frac{1}{20}\sin(2t) + \frac{1}{40}\cos(2t)$$

Final “particular” solution
$$y(t) = y_h(t) + Y(t)$$

$$y(t) = c_1 e^{6t} + c_2 e^{-2t} - \frac{1}{20}\sin(2t) + \frac{1}{40}\cos(2t)$$

Main trick here is to try to get the correct Ansatz for $Y(t)$.
See Worksheet posted in Canvas. Solutions here:

1. $A\sin(t) + B\cos(t)$
2. Ae^t
3. $does\ not\ apply, need\ variation\ of\ parameters$
4. At^2e^{-t}
5. Ae^{-2t}
6. $A\sin(t) + B\cos(t)$
7. $Ae^{-t}\sin(2t) + Be^{-t}\cos(2t)$
8. $Ate^{-5t}\sin\left(\frac{1}{2}t\right) + Bte^{-5t}\cos\left(\frac{1}{2}t\right)$
9. $(At^3 + Bt^2 + Ct + D)e^t(t^2) = At^5e^t + Bt^4e^t + Ct^3e^t + Dt^2e^t$
10. $does\ not\ apply$
11. $At\sin(2t) + Bt\cos(2t)$
12. $A\sin(t) + B\cos(t) + Cte^{2t}$
13. $At^4 + Bt^3 + Ct^2 + Dt + E + Fe^{-t}\sin(\sqrt{3}t) + Ge^{-t}\cos(\sqrt{3}t)$
14. $A + Bte^t + Ce^t$
15. $At + B + Cet^t\sin(t) + De^t\cos(t)$
16. $Ae^{-2t} + B\sin(4t) + C\cos(4t) + Dte^{-2t}\sin(4t) + Ete^{-2t}\cos(4t)$
17. $Ae^t + Bte^{-t}$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Variation of Parameters

y_1, y_2 form a fundamental solution set for the second order differential equation, and $W(x)$ is the Wronskian derived from those solutions. $g(x)$ is the forcing function

$$y'' + p(x)y' + q(x)y = g(x)$$

Then the non-homogeneous solution $Y(x) =$

$$Y(x) = -y_1 \int \frac{y_2 g(x)}{W(x)} dx + y_2 \int \frac{y_1 g(x)}{W(x) dx}$$

In some cases, you may obtain terms that have the same form as the homogeneous solutions. You can combine them into one term since these will only modify the coefficient and not add any unique function information.

Example.

$$ty'' - (t+1)y' + y = t^2$$

Homogeneous solutions are: $y_1 = e^t, y_2 = t + 1$

Find the Wronskian: verify that the solutions form a fundamental set, and give us the value of the Wronskian for our equation.

$$W(t) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = e^t - e^t(t+1) = e^t - te^t - e^t = -te^t$$

Put equation in standard form to obtain $g(x)$

$$ty'' - (t+1)y' + y = t^2$$

$$y'' - \frac{t+1}{t} y' + \frac{1}{t} y = t$$

$$g(t) = t$$

$$Y(x) = -y_1 \int \frac{y_2 g(x)}{W(x)} dx + y_2 \int \frac{y_1 g(x)}{W(x) dx}$$

$$Y(t) = -e^t \int \frac{(t+1)t}{te^t} dt + (t+1) \int \frac{e^t(t)}{te^t} dt = -e^t \int (t+1)e^{-t} dt + (t+1) \int 1 dt =$$

$$-e^t \left[-e^{-t}(t+1) - \int -e^{-t} dt \right] + (t+1)t = -e^t[-e^{-t}(t+1) - e^{-t}] + (t^2 + t)$$

$$u = t + 1, dv = e^{-t}$$
$$du = 1dt, v = -e^{-t}$$

$$(t+1) + 1 + (t^2 + t) = t^2 + 2t + 2$$

Final solution:

$$y(t) = c_1 e^t + c_2(t+1) + t^2 + 2t + 2$$

$$y(t) = c_1 e^t + c_2(t+1) + t^2 + 2(t+1) = c_1 e^t + c_3(t+1) + t^2$$

Where

$$c_3 = c_2 + 2$$

Redoing our first example with variation of parameters.

$$y'' - 4y' - 12y = 3e^{5t}$$

1) Solve the homogeneous solutions.

$$y'' - 4y' - 12y = 0$$
$$k^2 - 4k - 12 = 0$$
$$(k-6)(k+2) = 0$$

$$k = 6, -2$$
$$y_h(t) = c_1 e^{6t} + c_2 e^{-2t}$$
$$y_1 = e^{6t}, y_2 = e^{-2t}$$

2) Show that they are a fundamental set of solutions, and find the value of the Wronskian.

$$W(t) = \begin{vmatrix} e^{6t} & e^{-2t} \\ 6e^{6t} & -2e^{-2t} \end{vmatrix} = -2e^{4t} - 6e^{4t} = -8e^{4t}$$

3) Place the equation in standard form.
   This one is already in standard form.

4) Construct $Y(t)$ using the formula.

$$Y(t) = -e^{6t}\int \frac{e^{-2t}(3e^{5t})}{-8e^{4t}}dt + e^{-2t}\int \frac{e^{6t}(3e^{5t})}{-8e^{4t}}dt = \frac{3}{8}e^{6t}\int e^{-t}dt - \frac{3}{8}e^{-2t}\int e^{7t}dt =$$

$$\frac{3}{8}e^{6t}(-e^{-t}) - \frac{3}{8}e^{-2t}\left(\frac{1}{7}e^{7t}\right) = -\frac{3}{8}e^{5t} - \frac{3}{56}e^{5t} = -\frac{3}{7}e^{5t}$$

   This is the same solution we got doing this the other way, but undetermined coefficients.

Let's look at the second undetermined coefficient example:

$$y'' - 4y' - 12y = \sin(2t)$$

$$Y(t) = -e^{6t}\int \frac{e^{-2t}(\sin(2t))}{-8e^{4t}}dt + e^{-2t}\int \frac{e^{6t}(\sin(2t))}{-8e^{4t}}dt$$

$$= \frac{1}{8}e^{6t}\int e^{-6t}\sin(2t)\,dt - \frac{1}{8}e^{-2t}\int e^{2t}\sin(2t)\,dt$$

Both integrals require integration by parts and they are looping integrals.

We will return to problems with forcing functions particularly when we look at spring problems and circuits.

Next week is the method of reduction of order.