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Second Order Equations (Ch 5)
Constant Coefficients, Cauchy-Euler

Second order equations:

$$a(x)y'' + b(x)y' + c(x)y = F(x)$$

$F(x)$ is often referred to as the forcing function.

When $F(x) = 0$, the equation is said to be homogeneous.
First we will learn to solve the homogeneous equations, and then later, the non-homogeneous case.

Constant Coefficient case:

$$a(x) = a, b(x) = b, c(x) = c$$

Ex.

$$4y'' + 4y' + y = 0$$

Cauchy-Euler case:

$$a(x) = ax^2, b(x) = bx, c(x) = c$$

Ex.

$$x^2y'' - 3xy' + 5y = 0$$

For the constant coefficient case: we have derivatives that differ only by a constant and cancel out. What kind of function could that possibly be?

The solution must be of the form $y = e^{kx}$.

For the Cauchy-Euler case: we have derivative that differ from each other by a multiple of x. The base function guess is $y = x^n$

For the constant coefficient case:

$$ay'' + by' + cy = 0$$

Assume the solution is of the form: $y = e^{kx}$ (technically $y = c_1e^{kx}$)

$$a\left(k^2e^{kx}\right) + b\left(ke^{kx}\right) + ce^{kx} = 0$$

$$e^{kx}(ak^2 + bk + c) = 0$$

That means that the polynomial in k determines the solution.

We can use the quadratic formula to solve this, or if it's factorable, we can do that.

For the Cauchy-Euler case,

$$ax^2y'' + bxy' + cy = 0$$

Assume the solution is of the form $y = x^n$ (technically, $y = c_1x^n$)

$$ax^2\big(n(n-1)\big)x^{n-2} + bx(nx^{n-1}) + cx^n = 0$$

$$an(n-1)x^n + bnx^n + cx^n = 0$$

$$x^n(an(n-1) + bn + c) = 0$$

$$an^2 - an + bn + c = 0$$
$$an^2 + (b-a)n + c = 0$$

Both of these methods come down to solving a quadratic.

First order equations had only one constant of integration and only one solution. For second order differential equations we end up with two coefficients to solve for, and therefore any initial value problem (or boundary value problem) must have two conditions in order to solve for both coefficients.

And we will need two independent functions to find all possible solutions.

For higher order problems , the math basically works out the same way. We assume the appropriate solutions type (based on whether the coefficient are constant or Cauchy-Euler), and then we get a polynomial that we have to solve.

Characteristic equation: the constant coefficient equation produces a polynomial that is called a characteristic equation: $ak^2 + bk + c = 0$. The Cauchy-Euler polynomial is not called characteristic. It's called auxiliary. $an^2 + (b-a)n + c = 0$.

Example.

$$y'' + 11y' + 24y = 0, y(0) = 0, y'(0) = -7$$

Replace $y'' \to k^2, y' \to k, y \to 1$ to get characteristic equation

$$k^2 + 11k + 24 = 0$$

$$(k+3)(k+8) = 0$$

$$k = -3, -8$$

Proposed general solution for the differential equation:

$$y = c_1e^{-3x} + c_2e^{-8x}$$

Linear combination of the solutions for the differential equation.

$$0 = c_1 + c_2$$

$$y' = -3c_1e^{-3x} - 8c_2e^{-8x}$$
$$-7 = -3c_1 - 8c_2$$

$$c_1 + c_2 = 0$$
$$-3c_1 - 8c_2 = -7$$

$$c_1 = -c_2$$
$$-3(-c_2) - 8c_2 = -7$$
$$3c_2 - 8c_2 = -7$$
$$-5c_2 = -7$$

$$c_2 = \frac{7}{5}$$

$$c_1 = -\frac{7}{5}$$

$$y(x) = -\frac{7}{5}e^{-3x} + \frac{7}{5}e^{-8x}$$

My particular solution to the initial value problem.

Example.

$$x^2y'' - 4xy' + 6y = 0$$

Replace $x^2y'' \to n(n-1), xy' \to n, y \to 1$

$$n(n-1) - 4n + 6 = 0$$

$$n^2 - n - 4n + 6 = 0$$
$$n^2 - 5n + 6 = 0$$

$$(n-2)(n-3) = 0$$
$$n = 2,3$$

$$y = c_1x^2 + c_2x^3$$

Often in Cauchy-Euler IVPs, they don't use 0, but will use some other number like 1.

Linear independence is a concept from linear algebra. In this context, we want to be able to generate all the possible solutions of our differential equation. It's not just enough for the proposed solutions to both satisfy the differential equation. They also must be independent.

Suppose $y_1 = 2x^2, y_2 = 4x^2$

These are both solutions to the differential equation, but they are not linearly independent. When we have only two functions, this basically comes down to is one function a multiple of the other function.

Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $W(x) \neq 0$ (for all x), then the set of solutions is independent. If the W(x)=0 for all x, then the set of solutions is dependent.

$$W = \begin{vmatrix} 2x^2 & 4x^2 \\ 4x & 8x \end{vmatrix} = 2x^2(8x) - 4x(4x^2) = 16x^3 - 16x^3 = 0$$

Compared to the x^2, x^3 solutions.

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4$$

This is not 0 (for most values of x).
These two solutions are independent. Thus we can use them to find any possible solution to the a set of initial conditions.

For higher order problems, you will need to take a bigger determinant.

For third order:

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1 \begin{vmatrix} y_2{}' & y_3{}' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1{}' & y_3{}' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1{}' & y_2{}' \\ y_1'' & y_2'' \end{vmatrix}$$

$$= y_1[y_2'(y_3'') - y_2''(y_3')] - y_2[y_1'(y_3'') - y_3'(y_1'')] + y_3[y_1'(y_2'') - y_1''(y_2')]$$

Abel's Theorem

For a second order equation in standard for form: $y'' + p(x)y' + q(x)y = 0$

$$W(x) = c_0 e^{\int -p(x)dx}$$

Don't confuse this with the integrating factor from linear first order equations. They are similar, but note the sign change!!!

$$x^2y'' - 4xy' + 6y = 0$$

In standard form:

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$$

$$p(x) = -\frac{4}{x}$$

$$W(x) = c_0 e^{-\int -\frac{4}{x}dx} = c_0 e^{\int \frac{4}{x}dx} = c_0 e^{4\ln(x)} = c_0 e^{\ln(x^4)} = c_0 x^4$$

For constant coefficients:

$$y'' + 11y' + 24y = 0$$

$$p(x) = 11$$

$$W(x) = c_0 e^{-\int p(x)dx} = c_0 e^{\int -11dx} = c_0 e^{-11x}$$
$$W(x) = c_0 e^{-11x}$$

$$W = \begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix} = -8e^{-11x} - (-3e^{-11x}) = -5e^{-11x}$$

When we have enough solutions that are linearly independent, that is called a fundamental set of solutions.

For higher order problems, Abel's theorem is similar in this sense:

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

The same formula applies for the Wronskian: in standard form, the next derivative one down from the highest derivative determines the p(x) function.

Consider the equation:

$$y'' + (3x - 1)y' + 6y = 0$$

Even though I can't solve the equation, I can find the value of Wronskian.

$$p(x) = 3x - 1$$

$$W(x) = c_0 e^{-\int 3x-1 dx} = c_0 e^{\left(-\frac{3}{2}x^2 + x\right)}$$

But it may help me to guess possible solutions to try.

Repeated roots.

Consider the differential equation

$$y'' + 4y' + 4y = 0$$

$$k^2 + 4k + 4 = 0$$
$$(k + 2)^2 = 0$$

$$k = -2$$

$$y_1 = c_1 e^{-2x}$$

The trick to make the second solution (for the constant coefficient case) is to multiply the solution by x.

$$y_2 = c_2 x e^{-2x}$$

General solution:

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y_2' = c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y_2'' = -2c_2 e^{-2x} - 2c_2 e^{-2x} + 4c_2 x e^{-2x} = -4c_2 e^{-2x} + 4c_2 x e^{-2x}$$

$$y'' + 4y' + 4y = 0$$

$$-4e^{-2x} + 4xe^{-2x} + 4(e^{-2x} - 2xe^{-2x}) + 4(xe^{-2x}) =?\, 0$$

$$-4e^{-2x} + 4xe^{-2x} + 4e^{-2x} - 8xe^{-2x} + 4xe^{-2x} =?\, 0$$

They both work in the equation. Now, is the Wronskian non-zero:

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-2x}(e^{-2x} - 2xe^{-2x}) - (-2e^{-2x})(xe^{-2x}) =$$

$$e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$$

This is a fundamental set.
If you have a repeated root (from a perfect square polynomial), one solution is e^{kx} and the second solution is xe^{kx}.

For Cauchy-Euler, we need a different trick. The trick we use is to multiply by ln(x).

$$x^2y'' + 5xy' + 4y = 0$$

$$n(n-1) + 5n + 4 = 0$$
$$n^2 - n + 5n + 4 = 0$$
$$n^2 + 4n + 4 = 0$$
$$(n+2)^2 = 0$$

$$n = -2$$
$$y_1 = c_1x^{-2}$$

$$y_2 = c_2x^{-2}\ln(x)$$

General solution:

$$y(x) = c_1x^{-2} + c_2x^{-2}\ln(x)$$

In higher order problems, say, a constant coefficient case, where k is a repeated root more than 2 times, Just keep multiplying by x to get additional solutions: $e^{kx}, xe^{kx}, x^2e^{kx}, etc.$

Complex root cases.

Start with pure imaginary roots.

$$y'' + y = 0$$

$$k^2 + 1 = 0$$
$$k = \pm i$$

What kind of a function has the property that $y'' = -y$

$$y = \sin(x)$$
$$y' = \cos(x)$$
$$y'' = -\sin(x)$$

Or

$$y = \cos(x)$$
$$y' = -\sin(x)$$
$$y'' = -\cos(x)$$

The functions that satisfy this relationship $y'' = -y$ are $y = c_1 \sin(x)\,, y = c_2 \cos(x)$

That is equivalent to e^{ix} *and* e^{-ix}

How does that work??

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

Euler's Theorem.

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sinh(x) = \frac{(e^{x} - e^{-x})}{2}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

Fully complex solutions are $k = \lambda \pm \mu i$

$$e^{(\lambda+\mu i)x} + e^{(\lambda-\mu i)x} = e^{(\lambda x+\mu ix)} + e^{(\lambda x-\mu ix)} = e^{\lambda x}e^{\mu ix} + e^{\lambda x}e^{\mu ix}$$

$$= e^{\lambda x}\left(e^{(\mu x)i} + e^{-\mu xi}\right) = e^{\lambda x}(2)\left(\frac{e^{(\mu x)i} + e^{-\mu xi}}{2}\right) = 2e^{\lambda x}(\cos(\mu x))$$

If I choose a constant multiplier, c_1, I can use that to absorb that extra 2.

$$y_1 = c_1 e^{\lambda x}\cos(\mu x)$$

If I subtract the two expressions I can obtain sine

$$e^{(\lambda+\mu i)x} - e^{(\lambda-\mu i)x} = e^{(\lambda x+\mu ix)} - e^{(\lambda x-\mu ix)} = e^{\lambda x}e^{\mu ix} - e^{\lambda x}e^{\mu ix}$$

$$= e^{\lambda x}\left(e^{(\mu x)i} - e^{-\mu xi}\right) = e^{\lambda x}(2i)\left(\frac{e^{(\mu x)i} + e^{-\mu xi}}{2i}\right) = 2ie^{\lambda x}(\sin(\mu x))$$

2nd solution is, choosing another constant to absorb the 2i, to get

$$y_2 = c_2 e^{\lambda x}\sin{(\mu x)}$$

In general, if you have a complex root to your characteristic equation, then the general solution is of the form

$$k = \lambda \pm \mu i$$

$$y(x) = c_1 e^{\lambda x}\cos(\mu x) + c_2 e^{\lambda x}\sin(\mu x)$$

In the Wronskian, you can show that these are independent.

Example.

$$y'' - 4y' + 9y = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$k^2 - 4k + 9 = 0$$

$$k = \frac{4 \pm \sqrt{16 - 4(1)(9)}}{2(1)} = \frac{4 \pm \sqrt{-20}}{2} = \frac{4 \pm 2\sqrt{5}i}{2} = 2 \pm \sqrt{5}i$$

$$\lambda = 2, \mu = \sqrt{5}$$

$$y(x) = c_1 e^{2x}\cos\left(\sqrt{5}x\right) + c_2 e^{2x}\sin\left(\sqrt{5}x\right)$$

For Cauchy-Euler case.

Think about pure imaginary case.

$$n = \pm i$$

$$x^i = \left(e^{lnx}\right)^i = e^{i\ln x} = \cos(\ln x) + i\sin(\ln x)$$

$$n = \lambda \pm \mu i$$

$$x^{\lambda+\mu i} = \left(e^{\ln x}\right)^{\lambda+\mu i} = e^{\lambda\ln x+\mu i\ln x} = e^{\lambda\ln x}e^{(\mu\ln x)i} = e^{\lambda\ln x}[\cos(\,\mu\ln x) + i\sin(\mu\ln x)]$$

Two solutions for the general solution are:

$$y(x) = c_1 e^{\lambda \ln x} \cos(\mu \ln x) + c_2 e^{\lambda \ln x} \sin(\mu \ln x)$$

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x)$$

You can use the Wronskian to show that these are independent and form a fundamental set.

All complex roots have to come in pairs.

Next time, we'll start looking at nom-homogeneous cases.