MTH 267, Homework #10, Fall 2022 Name ____________________________________

Instructions: Work problems on a separate sheet of paper and attach work to this page. You should show all work to receive full credit for problems. Checking your work with computer algebra systems is fine, but that doesn't count as "work" since you won't be able to use CAS programs on exams or quizzes. Graphs and longer answers that won't fit here, indicate which page of the work the answer can be found on and be sure to clearly indicate it on the attached pages.

1. Use Laplace transforms to solve the given initial value problems.
   a. $y'' - y' - 6y = 0, y(0) = 1, y'(0) = -1$
   b. $y^{IV} - y = 0, y(0) = 1, y'(0) = 0, y''(0) = 1, y'''(0) = 0$
   c. $y'' + 2y' + y = 4e^{-t}, y(0) = 2, y'(0) = -1$
   d. $y'' + 4y = \begin{cases} 1, 0 \le t < \pi \\ 0, \pi \le t < \infty \end{cases}, y(0) = 1, y'(0) = 0$

2. Sketch the graph of the function given in terms of the step function $u_c(t) = \begin{cases} 0, t < c \\ 1, t \ge c \end{cases}$ on the interval $t > 0$. [Note: Notations are mixed here deliberately. Express your solutions in either $u_c(t)$ notation or $u(t-c)$ notation.]
   a. $g(t) = u_1(t) + 2u(t-3) - 6u_4(t)$
   b. $g(t) = f(t-1)u_2(t), where\ f(t) = 2t$
   c. $g(t) = f(t-3)u(t-3), where\ f(t) = \sin t$

   Note: The alternative notation for $u_c(t)$ is $u(t-c)$. You may use either here and throughout.

3. Sketch the graph of the function, then express each function in terms of the unit step function.
   a. $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$
   b. $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & 2 \le t < 5 \\ 7-t, & 5 \le t < 7 \\ 0, & t \ge 7 \end{cases}$

4. Find the Laplace transform of the function $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$. It may be useful to rewrite this in terms of the unit step function and then use that transformation rule, but you can also use the definition of the transform if you prefer.

5. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of each of the following functions. Use the table of transforms. You will have to do partial fraction decomposition on some of these, or complete squares on others.
   a. $F(s) = \frac{3}{s^2+4}$
   b. $F(s) = \frac{8s^2-4s+12}{s(s^2+4)}$
   c. $F(s) = \frac{3s}{s^2-s-6}$
   d. $F(s) = \frac{2s-3}{s^2+2s+10}$

6. The following two functions involve convolutions. Use the table of Laplace transforms to find the transform of a, and the inverse transform of b.
   a. $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$
   b. $F(s) = \frac{s}{(s+1)(s^2+4)}$

7. Express the solution of the given initial value problem in terms of a convolution integral.
   e. $y'' + 2y' + 2y = \sin \alpha t, y(0) = 0, y'(0) = 0$
   f. $y^{IV} - y = g(t), y(0) = y'(0) = y''(0) = y'''(0) = 0$