MTH 267, Final Exam, Fall 2022 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Solve the linear ODE $2y' + y = 3t$ using an integrating factor. (12 points)

$$y' + \tfrac{1}{2}y = \tfrac{3}{2}t \qquad \mu = e^{\int 1/2\,dt} = e^{1/2 t}$$

$$e^{1/2t}y' + \tfrac{1}{2}e^{1/2t}y = \tfrac{3}{2}te^{1/2t}$$

$$\int (e^{1/2t}y)' = \int \tfrac{3}{2}te^{1/2t}\,dt$$

$$e^{1/2t}y = 3te^{1/2t} - 6e^{1/2t} + C$$

$$\boxed{y = 3t - 6 + Ce^{-1/2t}}$$

	u	dv
$+$	$3/2\,t$	$e^{1/2t}$
$-$	$3/2$	$2e^{1/2t}$
		$4e^{1/2t}$

2. Use Runge-Kutta on [redacted], $y' = xy$, [redacted], $y(0) = 1$ for one step [redacted]) with step size $h = 0.2$. (15 points)

$$K_{01} = 0(1) = 0$$

$$y_{01} = 1 + \tfrac{1}{2}(0.2)(0) = 1$$

$$k_{02} = 0.1(1) = 0.1$$

$$y_{02} = 1 + \tfrac{1}{2}(0.2)(0.1) = 1.01$$

$$k_{03} = 0.1(1.01) = 0.101$$

$$y_{03} = 1 + (0.2)(0.101) = 1.0202$$

$$k_4 = 0.2(1.0202) = 0.20404$$

$$y_1 = 1 + \tfrac{0.2}{6}(1 + 2(1) + 2(1.01) + 1.0202) = 1.020201$$

3. Determine if the set of solutions $f_1(t) = 2t - 3, f_2(t) = t^3 + 1, f_3(t) = 2t^2 - t, f_4(t) = t^2 + t + 1$, are linearly independent. (15 points)

$$\begin{vmatrix} 2t-3 & t^3+1 & 2t^2-t & t^2+t+1 \\ 2 & 3t^2 & 4t-1 & 2t+1 \\ 0 & 6t & 4 & 2 \\ 0 & 6 & 0 & 0 \end{vmatrix} = 6\begin{vmatrix} 2t-3 & 2t^2-t & t^2+t+1 \\ 2 & 4t-1 & 2t+1 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= 6\left[(2t-3)\begin{vmatrix} 4t-1 & 2t+1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 2t^2-t & t^2+t+1 \\ 4 & 2 \end{vmatrix}\right] = 6\Big[(2t-3)\big(\overbrace{(4t-1)2 - 4(2t+1)}^{8t-2-8t-4}\big) - 2\big(\underbrace{(2t^2-t)(2) - 4(t^2+t+1)}_{4t^2-2t-4t^2-4t-4 \;=\; -6t-8}\big)\Big] =$$

$$6\big[\underbrace{(2t-3)(-6)}_{-12t+18} - \underbrace{2(-6t-8)}_{-12t+16}\big] = 6\big[-24t+34\big] = -144t+204$$

independent, not identically 0

4. Solve the homogeneous second order ODE $y'' + 4y' + 3y = 0, y(0) = 2, y'(0) = -1$. (12 points)

$r^2 + 4r + 3 = 0 \quad (r+3)(r+1) = 0 \quad r = 1, -3$

$y(t) = c_1 e^{-t} + c_2 e^{-3t}$

$2 = c_1 + c_2$

$y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

$-1 = -c_1 - 3c_2$

$\cancel{c_1} + c_2 = 2$

$\cancel{-c_1} - 3c_2 = -1$

$-2c_2 = 1$

$c_2 = -1/2$

$c_1 - \frac{1}{2} = 2$

$c_1 = 5/2$

$y(t) = 5/2\, e^{-t} - \frac{1}{2} e^{-3t}$

5. Solve the differential equation $2y^{IV} + 13y''' - 5y'' + 13y' - 7y = 0$ (15 points)

$2r^4 + 13r^3 - 5r^2 + 13r - 7 = 0$

zeros at $x = -7, r = 1/2$ factors $(r+7)(2r-1)$ divide out to get $x^2 + 1$

$(r+7)(2r-1)(r^2+1) = 0 \quad r = 7, 1/2, \pm i$

use a graph to get real roots (rational ones)

$y(t) = c_1 e^{-7t} + c_2 e^{1/2 t} + c_3 \cos t + c_4 \sin t$

6. A mass weighing 64 lbs stretches a spring 9 in. The mass is attached to a viscous damper with a damping constant of 6 lbs·sec/ft.
   a. If the mass is set in motion from its equilibrium position with a downward velocity of 1 in/sec, find its position y at any time t. (16 points)

$My'' + \gamma y' + ky = 0 \qquad y(0) = 0 \qquad y'(0) = -1/12$

$F = ma$

$64 = m(32)$

$m = 2 \qquad \gamma = 6$

$2y'' + 6y' + \frac{256}{3} y = 0$

$y'' + 3y' + \frac{128}{3} y = 0$

$F = 64 = ky$

$64 = k\left(\frac{3}{4}\right) \Rightarrow k = \frac{256}{3}$

$3y'' + 9y' + 128y = 0 \qquad 3r^2 + 9r + 128 = 0$

   b. Sketch the graph of the function. (5 points)

$r = \frac{-9 \pm \sqrt{81 - 4(3)(128)}}{2(3)} = \frac{-9 \pm \sqrt{1455}\, i}{6}$

$y(t) = c_1 e^{-3/2 t} \cos\left(\frac{\sqrt{1455}}{6} t\right) + c_2 e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{1455}}{6} t\right)$

↓

6 continued

$y(0) = 0 = c_1(1)(1) + \cancel{c_2(1)(0)} \qquad c_1 = 0$

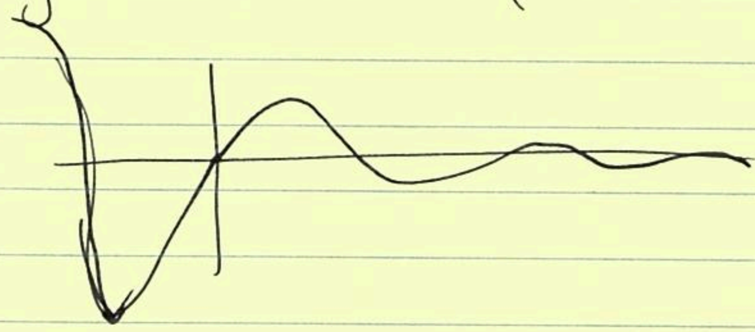
$y(t) = c_2 e^{-3/2t} \sin\left(\frac{\sqrt{1455}}{6} t\right)$

$y'(t) = -3/2\, c_2 e^{-3/2t} \sin\left(\frac{\sqrt{1455}}{6} t\right) + c_2 e^{-3/2t} \cos\left(\frac{\sqrt{1455}}{6} t\right) \cdot \frac{\sqrt{1455}}{6}$

$-1/12 = y'(0) = \cancel{-3/2\, c_2(1)(0)} + c_2(1)(1)\frac{\sqrt{1455}}{6}$

$-\frac{1}{2\sqrt{1455}} = c_2$

$y(t) = \frac{-1}{2\sqrt{1455}} e^{-3/2t} \sin\left(\frac{\sqrt{1455}}{6} t\right)$



c. Is it undamped, underdamped, critically damped or overdamped? How can you tell? (5 points)

underdamped $e^{-3/2t}$

d. Does the system experience resonance, beats or neither? (5 points)

neither (no forcing)

e. What is the transient solution? (3 points)

$y(t) = \frac{-1}{2\sqrt{1455}} e^{-3/2t} \sin\left(\frac{\sqrt{1455}}{6}t\right)$ the whole thing

f. What is the steady state solution? (3 points)

$y = 0$

7. Use the **definition** of Laplace transforms to find the transform F(s) for the function $f(t) = te^t$. You may check your answers with the included table, but you must show the integration work to receive credit. (15 points)

$$\int_0^\infty e^{-st} t e^t \, dt = \int_0^\infty t e^{-t(s-1)} \, dt$$

u	dv
$+\ t$	$e^{-t(s-1)}$
$-\ 1$	$\frac{-1}{s-1} e^{-t(s-1)}$
	$\frac{1}{(s-1)^2} e^{-t(s-1)}$

$$\cancel{\frac{-1}{s-1} t e^{-t(s-1)}} - \frac{1}{(s-1)^2} e^{-t(s-1)} \Big|_0^\infty =$$

$$0 - 0 + 0 + \frac{1}{(s-1)^2} = \frac{1}{(s-1)^2}$$

8. Find the Laplace Transforms for the following functions. You may use the table of formulas. (5 points each)
   a. $\mathcal{L}\{3\cos(7t)\}$ $\qquad \frac{3s}{s^2+49}$

   b. $\mathcal{L}\left\{-\frac{1}{2}t^4e^{-2t}\right\}$ $\qquad -\frac{1}{2}\frac{4!}{(s+2)^5} = \frac{-12}{(s+2)^5}$

   c. $\mathcal{L}\{6\sinh(\pi t)\}$ $\qquad \frac{6\pi}{s^2-\pi^2}$

   d. $\mathcal{L}\{3t\cosh(t)\}$ $\qquad \frac{3(s^2+1)}{(s^2-1)^2}$

   e. $\mathcal{L}\{4u_3(t) - 2u_5(t)\}$ $\qquad \frac{4e^{-3s}}{s} - \frac{2e^{-5s}}{s}$

9. Find the inverse Laplace transform of the following functions. (5 points each)
   a. $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4}\right\} = \frac{2s}{s^2+4} - \frac{1}{s^2+4} = 2\cos 2t - \frac{1}{2}\sin 2t$

   b. $\mathcal{L}^{-1}\left\{\frac{4s}{s^2-6s+7}\right\} = \frac{4(s-3)+12}{(s-3)^2-2}$ $\quad 4e^{+3t}\cosh\sqrt{2}t + \frac{12}{\sqrt{2}}e^{3t}\sinh(\sqrt{2}t)$

   $(s^2-6s+9)-2$ $\qquad b=\sqrt{2}$

   $(s-3)^2-2$ $\qquad a=3$

   c. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$

   $= u_2(t)\cdot t = u(t-2)\cdot t$

   $3s-7 = 3(s+2)-6-7$

   d. $\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+4s+8}\right\} = \frac{3(s+2)}{(s+2)^2+4} - \frac{13}{(s-2)^2+4} \rightarrow 3e^{-2t}\cos 2t - 13/2\, e^{-2t}\sin 2t$

   $(s^2+4+4)+4$ $\qquad a=-2$

   $(s+2)^2+4$ $\qquad b=2$

   e. $\mathcal{L}^{-1}\left\{\frac{10}{(s-4)^7}\right\}$ $\qquad \frac{10}{720}t^6e^{4t} = \frac{1}{72}t^6e^{4t}$

   $n=6$

10. Solve the differential equations using Laplace Transforms. (15 points each)

a. $y'' + 6y' - 7y = te^t, y(0) = 0, y'(0) = 2$

$$s^2Y(s) - \cancel{s(0)} - 2 + 6sY(s) - \cancel{6(0)} - 7Y(s) = \frac{1}{(s-1)^2}$$

$$Y(s)\left[s^2+6s-7\right] - 2 = \frac{1}{(s-1)^2}$$

$$Y(s)(s^2+6s-7) = \frac{1+2(s-1)^2}{(s-1)^2} = \frac{1+2(s^2+2s+1)}{(s-1)^2} = \frac{1+2s^2-4s+2}{(s-1)^2} =$$

$$Y(s) = \frac{2s^2-4s+3}{(s-1)^2(s^2+6s-7)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{Cs+D}{\underset{(s+7)(s-1)}{s^2+6s-7}}$$

$$\frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+7} + \frac{D}{(s-1)^3}$$

$$A(s-1)^2(s+7) + B(s-1)(s+7) + C(s-1)^3 + D(s+7) = 2s^2-4s+3$$

$s=1$ $\quad D(8) = 2-4+3 \rightarrow D = 1/8$

$s=-7$ $\quad C(-8)^3 = 2(49)+4(7)+3 \rightarrow C = -129/512$

$\rightarrow$

b. $y'' - 5y' + 6y = f(t), y(0) = 1, y'(0) = 0$, for $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 4, & t \ge 3 \end{cases}$ $\quad 4u(t-3)$

$$s^2Y(s) - s(1) - 0 - 5sY(s) + 5(1) + 6Y(s) = \frac{4e^{-3s}}{s}$$

$$Y(s)(s^2-5s+6) - s + 5 = \frac{e^{-3s}}{s}\cdot 4$$

$$Y(s)(s^2-5s+6) = \frac{4e^{-3s}}{s} + s - 5 = \frac{4e^{-3s}+s^2-5s}{s}$$

$$Y(s) = \frac{4e^{-3s}+s^2-5s}{s(s-3)(s-2)} = e^{-3s}\left(\frac{A}{s}+\frac{B}{s-3}+\frac{C}{s-2}\right) + \left(\frac{D}{s}+\frac{E}{s-3}+\frac{F}{s-2}\right)$$

$$A(s-3)(s-2) + B(s)(s-2) + C(s)(s-3) = 4(e^{-3s})$$

$$D(s-3)(s-2) + E(s)(s-2) + F(s)(s-3) = s^2-5s$$

$s=0$ $\quad A(-3)(-2) = 4 \rightarrow 6A = 4 \rightarrow A = 2/3$ $\qquad D(-3)(-2) = 0 \quad D = 0$

$s=2$ $\quad C(2)(-1) = 4 \rightarrow C = -2$ $\qquad F(2)(-1) = 4-10 = -6 \quad F = 3$

$s=3$ $\quad B(3)(1) = 4 \rightarrow B = 4/3$ $\qquad E(3)(1) = 9-15 = -6 \quad E = -2$ $\rightarrow$

11. Find the inverse Laplace transform of $F(s) = \frac{5}{s^4+49s^2}$ using convolutions. (10 points)

$$\frac{5}{s^2(s^2+49)} = \frac{5}{s^2}\cdot\frac{1}{s^2+49}$$

$5t \qquad \frac{1}{7}\sin 7t$

$$f(t) = \int_0^t \frac{5}{7}(t-\tau)\sin 7\tau \, d\tau$$

or $\int_0^t \frac{5}{7}\tau \sin(t-\tau)\, d\tau$

10a. continued

$A(s^2-2s+1)(s+7) + B(s^2+7s-s-7) + C(s^3-3s^2+3s-1) + D(s+7)$

$A(s^3+7s^2-2s^2-14s+s+7) + B(s^2+6s-7) + C(s^3-3s^2+3s-1) + D(s+7)$

$A(s^3+5s^2-13s+7) + B(s^2+6s-7) + C(s^3-3s^2+3s-1) + D(s+7)$

$As^3+5As^2-13As+7A+Bs^2+6Bs-7B+Cs^3-3Cs^2+3Cs-C+Ds+7D$

$A+C=0$

$5A+B-3C=2$

$-13A+6B+3C+D=-4$

$7A-7B-C+7D=3$

$A=129/512$

$B=-1/64$

$C=-129/512$

$D=1/8$

$$Y(s) = \frac{129/512}{s-1} - \frac{1/64}{(s-1)^2} + \frac{1/8}{(s-1)^3} - \frac{129/512}{s+7}$$

$$y = \frac{129}{512}e^t - \frac{1}{64}te^t + \frac{1}{16}t^2e^t - \frac{129}{512}e^{-7t}$$

10b. continued

$$Y(s) = e^{-3s}\left(\frac{2/3}{s} + \frac{4/3}{s-3} - \frac{2}{s-2}\right) + \left(-\frac{2}{s-3} + \frac{3}{s-2}\right)$$

$$Y(t) = u(t-3)\left(2/3 + 4/3e^{3(t-3)} - 2e^{2(t-3)}\right) - 2e^{3t} + 3e^{2t}$$

12. Solve the systems of equations for the general solution below using eigenvalues. Be sure that your solutions are expressed only with real-valued functions. (14 points each)

a. $\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix} \vec{x}$

$(3-\lambda)(4-\lambda) - 6 = 0$

$\lambda^2 - 7\lambda + 12 - 6 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6, 1$

$\lambda = 6$

$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$ $\quad 3x_1 - 2x_2 = 0$

$3x_1 = 2x_2$

$x_1 = \frac{2}{3}x_2 \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$x_2 = x_2$

$\lambda = 1$

$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ $\quad 2x_1 + 2x_2 = 0$

$x_1 = -x_2 \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$x_2 = x_2$

$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t}$

b. $\vec{x}'(t) = \begin{pmatrix} -8 & 9 \\ -5 & 4 \end{pmatrix} \vec{x}$

$(-8-\lambda)(4-\lambda) + 45 = 0$

$\lambda^2 + 4\lambda - 32 + 45 = 0$

$\lambda^2 + 4\lambda + 13 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16-52}}{2} = \dfrac{-4 \pm 6i}{2} = -2 \pm 3i$

$\begin{pmatrix} -8-(-2+3i) & 9 \\ -5 & 4-(-2+3i) \end{pmatrix} = \begin{pmatrix} -8+2-3i & 9 \\ -5 & 4+2-3i \end{pmatrix} = \begin{pmatrix} -6-3i & 9 \\ -5 & 6-3i \end{pmatrix}$

$-5x_1 + (6-3i)x_2 = 0$

$x_1 = \frac{6-3i}{5} x_2$

$x_2 = x_2$

$\vec{v}_1 = \begin{pmatrix} 6-3i \\ 5 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 6+3i \\ 5 \end{pmatrix}$

$\begin{pmatrix} 6-3i \\ 5 \end{pmatrix}(\cos 3t + i \sin 3t) =$

$\begin{pmatrix} 6\cos 3t + 6i \sin 3t - 3i \cos 3t + 3 \sin 3t \\ 5 \cos 3t + 5i \sin 3t \end{pmatrix}$

$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} 6\cos 3t + 3 \sin 3t \\ 5 \cos 3t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 6 \sin 3t - 3\cos 3t \\ 5 \sin 3t \end{pmatrix}$

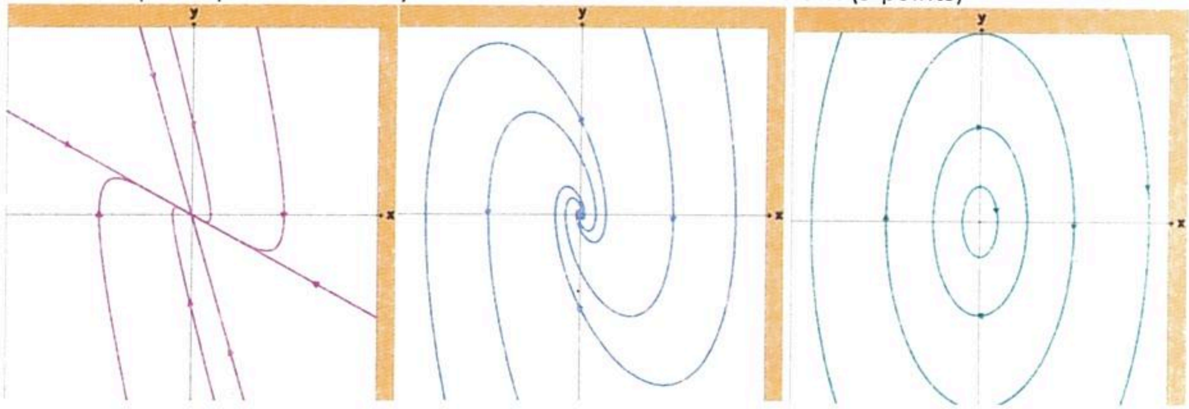
13. Solve the differential equation $\frac{dy}{dt} = \frac{t\sqrt{1-y^2}}{e^{2t}}$ using separation of variables. (15 points)

$\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int t e^{-2t}\, dt$

	u	dv
$+$	t	e^{-2t}
$-$	1	$-\frac{1}{2}e^{-2t}$
	0	$+\frac{1}{4}e^{-2t}$

$\arcsin y = -\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t} + C$

14. A series of phase portraits for a system of linear ODEs is shown below. (9 points)



A B C

Match the phase portrait with one of the differentia equations below that could be represented by the graph. Explain your reasoning.

a. $y'' + 3y = 0$

$r^2 + 3 = 0$

$r = \pm\sqrt{3}i$

C

sine/cosine only solution
stable orbits

b. $y'' + 4y' + 3y = 0$

$(r+3)(r+1) = 0$

$r = -3, -1$

A

exponential decay =
linear eigenvector

c. $y'' + 2y' + 5y = 0$

$r = \dfrac{-2 \pm \sqrt{4-20}}{2}$

$= \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

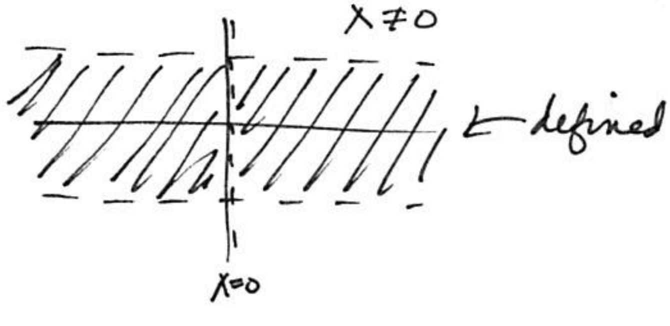
B

oscillating decay
= spiral

15. Use the Existence and Uniqueness Theorem to determine where the differential equation $y' = \frac{2\sqrt{1-y^2}}{x^2}$ is guaranteed to have a unique solution. Sketch the graph of the region. (12 points)

$1 - y^2 > 0 \rightarrow 1 > y^2 \quad -1 < y < 1$

$x \neq 0$



$f = \dfrac{2\sqrt{1-y^2}}{x^2} = \dfrac{2}{x^2}(1-y^2)^{1/2}$

$f'_y = \frac{2}{x^2}\left(\frac{1}{2}\right)(1-y^2)^{-1/2}(-2y)$

0 ok in original
but not here.

Modified Euler': $y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right]$

Runge-Kutta: $y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right)$,

$$k_{n1} = f(t_n, y_n), k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right),$$

$$k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right), k_{n4} = f(t_n + h, y_n + hk_{n3})$$