MTH 267, Exam #2, Fall 2022 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find the Wronskian for e^{-2t} and te^{-2t}. (12 points)

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) + 2e^{-2t}(te^{-2t})$$

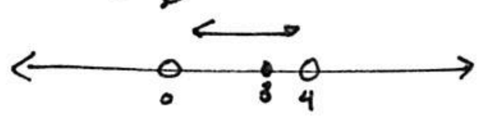
$$= e^{-4t} - \cancel{2te^{-4t}} + \cancel{2te^{-4t}} = e^{-4t}$$

2. Use Abel's Theorem to find the Wronskian for $t(t-4)y'' + 3ty' + 4y = 2$. What is the longest interval on which a solution to the IVP $y(3) = 0, y'(3) = -1$ is defined? (9 points)

$$y'' + \frac{3\cancel{t}}{\cancel{t}(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

$$W = e^{-\int \frac{3}{t-4}\,dt} = e^{-3\ln|t-4|} = \cancel{e}^{\cancel{\ln}(t-4)^{-3}} = (t-4)^{-3} = \frac{1}{(t-4)^3}$$

defined on $t \neq 0$, $t \neq 4$



$(0, 4)$ defined on for given IVP

3. Explain the conditions needed for resonance. Why is it a problem in a mechanical system? (8 points)

natural frequency and forcing frequency are identical.

This causes the amplitude of the oscillation to increase over time

this will lead to mechanical failure.

4. Solve the differential equations. (15 points each)

a. $y'' + 8y' + 16y = 0, y(0) = 1, y'(0) = 4$

$r^2 + 8r + 16 = 0$

$(r+4)^2 = 0$

$r = -4$ repeated

$y(t) = e^{-4t} + 8te^{-4t}$

$y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$

$y(0) = 1 = c_1 + c_2(0) \Rightarrow c_1 = 1$

$y'(t) = -4\overset{1}{\cancel{c_1}} e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$y'(0) = 4 = -4(1) + c_2(1) + 0$

$c_2 = 8$

b. $y''' - y'' + y' - y = 0, y(0) = 2, y'(0) = -1, y''(0) = -2.$

$r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + 1(r-1) = 0$

$(r-1)(r^2+1) = 0$

$r = 1, \pm i$

$y(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$

$y(0) = c_1 + c_2 + 0 = 2$

$y'(t) = c_1 e^t - c_2 \sin t + c_3 \cos t$

$y'(0) = c_1 + 0 + c_3 = -1$

$y''(t) = c_1 e^t - c_2 \cos t - c_3 \sin t$

$y''(0) = c_1 - c_2 + 0 = -2$

$c_1 + \cancel{c_2} = 0$

$c_1 - \cancel{c_2} = -2$

$2c_1 = -2 \rightarrow c_1 = -1$

$c_1 + c_2 = 0 \rightarrow -1 + c_2 = 0$

$c_2 = 1$

$c_1 + c_3 = -1$

$\cancel{-1} + c_3 = \cancel{-1} \rightarrow c_3 = 0$

$\boxed{y(t) = -e^t + \cos t}$

c. $t^2 y'' - 3ty' - 12y = 0$

$y = t^r$

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$t^2 r(r-1)t^{r-2} - 3trt^{r-1} - 12t^r = 0$

$t^r(r^2 - r) - 3rt^r - 12t^r = 0$

$t^r[r^2 - r - 3r - 12] = 0$

$r^2 - 4r - 12 = 0$

$(r-6)(r+2) = 0$

$r = 6, r = -2$

$y(t) = c_1 t^6 + c_2 t^{-2}$

or

$y(t) = c_1 t^6 + \frac{c_2}{t^2}$

5. Use the method of undetermined coefficients to solve $y'' + 2y' + y = 2e^{-t}$. (20 points)

$y''+2y'+y=0$

$r^2+2r+1=0$

$(r+1)^2=0$

$r=-1$ repeated

$y(t)=c_1e^{-t}+c_2te^{-t}$

$Y_p=At^2e^{-t}$

$Y_p'=2Ate^{-t}+-At^2e^{-t}$

$Y_p''=2Ae^{-t}+2Ate^{-t}-2Ate^{-t}+At^2e^{-t}=2Ae^{-t}-4Ate^{-t}+At^2e^{-t}$

$2Ae^{-t}-\cancel{4Ate^{-t}}+\cancel{At^2e^{-t}}+\cancel{4Ate^{-t}}-\cancel{2At^2e^{-t}}+\cancel{At^2e^{-t}}=e^{-t}$

$2Ae^{-t}=2e^{-t}$

$A=1$

$\boxed{y(t)=c_1e^{-t}+c_2te^{-t}+t^2e^{-t}}$

6. What Ansatz would you need to solve for the given forcing function $F(t)$ and the specified solutions $y_1(t),\ y_2(t)$ to the second order ODE. (6 points each)

	$y_1(t)$	$y_2(t)$	$F(t)$	Ansatz
a.	$\sin t$	$\cos t$	$3e^{2t}$	Ae^{2t}
b.	e^{-t}	e^{-4t}	$-5e^t\cos 2t$	$Ae^t\cos 2t + Be^t\sin 2t$
c.	e^t	e^{-2t}	t^2+7e^t	$At^2+Bt+C+Dte^t$
d.	$\sin 3t$	$\cos 3t$	$4\sin 3t$	$At\sin 3t + Bt\cos 3t$

7. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 2 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t. (Set up the ODE and state initial conditions only; you don't need to solve.) (1 2points)

$My''+\gamma y'+ky=F(t)$

$2y''+14y'+3y=0$

$y(0)=1$

$y'(0)=0$

$F=kx$

$6=k(2)$

$k=3$

8. Use reduction of order to find the second solution to the differential equation $(1-2x-x^2)y''+2(1+x)y'-2y=0$ given that $y_1=x+1$. (25 points)

$y_2 = v(x+1) = vx+v$, $y_2' = v'x+v+v'$

$y_2'' = v''x+v'+v'+v'' = v''x+2v'+v''$

$(1-2x-x^2)(v''x+2v'+v'') +$

$(2+2x)(v'x+v+v') - 2(vx+v) = 0$

$v''x+2v'+v''-2v''x^2-4v'x-2v''x-v''x^3-2v'x^2-x^2v''+2v'x+2v+2v'+2x^2v'+$

$2xv+2xv'-2vx-2v=0$

$v''(x+1-2x^2-2x-x^3-x^2)+v'(2-\cancel{4x}-\cancel{2x^2}+\cancel{2x}+2+\cancel{2x^2}+\cancel{2x})+v(\cancel{2}+\cancel{2x}-\cancel{2x}-\cancel{2})=0$

$v''(-x^3-3x^2-x+1)+v'(4)=0 \quad u=v',\ u'=v''$

$(-x^3-3x^2-x+1)u' = -4u$

$\frac{du}{u} = \frac{+4}{x^3+3x^2+x-1} = \frac{4}{(x+1)(x^2+2x-1)}$

$$\begin{array}{r} x^2+2x-1 \\ x+1\overline{)\,x^3+3x^2+x-1} \\ -\ x^3+x^2 \\ \hline 2x^2+x \\ -\ 2x^2+2x \\ \hline -x-1 \end{array}$$

$\frac{du}{u} = \frac{4}{(x+1)(x+2x-1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x-1}$

$Ax^2+2Ax-A+Bx^2+Cx+Bx+C=4$

$A+B=0$ $\quad -A+C=4$

$2A+B+C=0$

$A=-5/2,\ B=7/2,\ C=3/2$

$\longrightarrow$

9. Below are solutions to mechanical vibration problems. For each solution state whether the system experiences beats or resonance or neither. Also, state which part of the solution is the transient solution, and which is the steady state solution. (8 points each)

a. $y(t) = \underbrace{\frac{4}{3}\cos(11t) - \frac{7}{9}\sin(11t)}_{} + \underbrace{\frac{11}{10}\sin(10t)}_{}$ all terms are steady state, no transients

beats

b. $y(t) = \underbrace{3e^{-\frac{t}{10}}\sin(\sqrt{3}t) - 5e^{-\frac{t}{10}}\cos(\sqrt{3}t)}_{\text{transient}} + \underbrace{\cos(2t) - 12\sin(2t)}_{\text{steady state}}$

no beats or resonance

c. $y(t) = \underbrace{\frac{1}{4}\sin(2t)}_{} - \underbrace{\frac{1}{8}t\sin(2t) + \frac{1}{6}t\cos(2t)}_{}$ all terms are steady state

resonance

8 continued

$$\frac{du}{u} = \frac{-5/2}{x+1} + \frac{7/2 x + 3/2}{x^2+2x-1} = \frac{-5/2}{x+1} + \frac{1}{2}\left(\frac{7x+3}{x^2+2x-1}\right)$$

$$\frac{-5/2}{x+1} + \frac{7}{2}\left(\frac{x + 3/7}{x^2+2x-1}\right) =$$

$$q = x^2+2x-1$$
$$dq = 2x+2$$
$$\frac{1}{2}dq = x+1$$

$$\frac{-5/2}{x+1} + \frac{7}{2}\left(\frac{x+1 + 3/7 - 1}{x^2+2x-1}\right) =$$

$$\frac{-5/2}{x+1} + \frac{7}{2}\left(\frac{x+1}{x^2+2x-1}\right) + \frac{7}{2}\left(\frac{-4/7}{x^2+2x-1}\right) =$$

$$x^2+2x+1 -1-1$$

$$-\frac{7}{2}\cdot\frac{4}{7} = -2$$

$$\frac{-5/2}{x+1} + \frac{7}{2}\left(\frac{x+1}{x^2+2x-1}\right) - 2\left(\frac{1}{(x+1)^2-2}\right)$$

$$\frac{7}{2}\int \frac{1}{2}\frac{1}{q}\,dq$$

$$\int \frac{du}{u} = \int \frac{-5/2}{x+1} + \frac{7}{2}\left(\frac{x+1}{x^2+2x-1}\right) - 2\left(\frac{1}{(x+1)^2-2}\right) dx$$

$$\ln|u| = -5/2 \ln|x+1| + \frac{7}{4}\ln|x^2+2x-1| - \frac{2}{\sqrt{2}}\tanh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

or $$\ln|u| = -5/2\ln|x+1| + \frac{7}{4}\ln|x^2+2x-1| - \frac{1}{\sqrt{2}}\ln(-x+\sqrt{2}-1) - \frac{1}{\sqrt{2}}\ln(x+\sqrt{2}+1)$$

$$\ln|u| = \ln(x+1)^{-5/2} + \ln(x^2+2x-1)^{7/4} - \ln(-x+\sqrt{2}-1)^{+1/\sqrt{2}} - \ln(x+\sqrt{2}+1)^{+1/\sqrt{2}}$$

$$\ln|u| = \ln\left[\frac{(x^2+2x-1)^{7/4}\cdot A}{(x+1)^{5/2}(-x+\sqrt{2}-1)^{1/\sqrt{2}}(x+\sqrt{2}+1)^{1/\sqrt{2}}}\right]$$

$$u = \frac{A\,(x^2+2x-1)^{7/4}}{(x+1)^{5/2}(-x+\sqrt{2}-1)^{1/\sqrt{2}}(x+\sqrt{2}+1)^{1/\sqrt{2}}}$$

$$v = \int \frac{A\,(x^2+2x-1)^{7/4}}{(x+1)^{5/2}(-x+\sqrt{2}-1)^{1/\sqrt{2}}(x+\sqrt{2}+1)^{1/\sqrt{2}}}\,dx$$

$$y_2 = (x+1)\int \frac{A\,(x^2+2x-1)^{7/4}}{(x+1)^{5/2}(-x+\sqrt{2}-1)^{1/\sqrt{2}}(x+\sqrt{2}+1)^{1/\sqrt{2}}}\,dx$$

10. Consider the following second order differential equations that model mechanical vibrations. Determine whether the systems they model are undamped, underdamped, critically damped or overdamped. If the system is undamped, state the natural frequency of the system. If the system is underdamped, state the quasi-frequency. (8 points each)

a. $4y'' + y = 0, y(-2) = 1, y'(-2) = -1$

no damping (no y' term) natural frequency is $\frac{1}{2}$

$y'' + \frac{1}{4}y = 0 \quad r^2 + \frac{1}{4} = 0 \quad r = \pm\frac{1}{2}i$

b. $9y'' + 12y' + 4y = 0, y(0) = 2, y'(0) = -1$

$9r^2 + 12r + 4 = 0$

$$r = \frac{-12 \pm \sqrt{144 - 4(9)(4)}}{2(9)} = \frac{-12 \pm 0}{18} = -\frac{2}{3}$$ critically damped

11. Use the method of variation of parameters ~~and your solution from problem 1~~ to find the particular solution to $y'' - 4y' - 12y = te^{-2t}$. (25 points)

$r^2 - 4r - 12 = 0$ $\quad y_1 = e^{6t}, y_2 = e^{-2t}$

$(r-6)(r+2) = 0$

$r = 6, -2$ $\quad W = \begin{vmatrix} e^{6t} & e^{-2t} \\ 6e^{6t} & -2e^{-2t} \end{vmatrix} = -2e^{4t} - 6e^{4t} = -8e^{4t}$

$$Y_p = -e^{6t}\int \frac{e^{-2t} \cdot te^{-2t}}{-8e^{4t}}\,dt + e^{-2t}\int \frac{e^{6t} \cdot te^{-2t}}{-8e^{4t}}\,dt =$$

$$\frac{1}{8}e^{6t}\int \frac{te^{-4t}}{e^{-4t}}\,dt - \frac{1}{8}e^{-2t}\int \frac{te^{4t}}{e^{4t}}\,dt = \frac{1}{8}e^{6t}\int te^{-8t}\,dt - \frac{1}{8}e^{-2t}\int t\,dt$$

$$\frac{1}{8}e^{6t}\left(-\frac{1}{8}te^{-8t} - \frac{1}{64}e^{-8t}\right) - \frac{1}{16}e^{-2t} \cdot t^2$$

	u	dv
+	t	e^{-8t}
−	1	$-\frac{1}{8}e^{-8t}$
	0	$\frac{1}{64}e^{-8t}$
		$-\frac{1}{512}e^{-8t}$

$$-\frac{1}{64}te^{-2t} - \frac{1}{512}e^{-2t} - \frac{t^2}{16}e^{-2t}$$

can be absorbed into homogeneous solution

$$y(t) = c_1e^{6t} + c_2e^{-2t} - \frac{1}{64}te^{-2t} - \frac{1}{16}t^2e^{-2t}$$