MTH 267, Exam #1, Fall 2022 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Solve the first order linear ordinary differential equation $y' + \frac{2}{t}y = te^{-3t}$ by the method of integrating factors. (15 points)

$$\mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2y' + 2ty = t^3e^{-3t}$$

$$\int (t^2y)' = \int t^3e^{-3t}dt$$

$$t^2y = -\tfrac{1}{3}t^3e^{-3t} - \tfrac{1}{3}t^2e^{-3t} - \tfrac{2}{9}te^{-3t} - \tfrac{2}{27}e^{-3t} + C$$

$$y = -\tfrac{1}{3}te^{-3t} - \tfrac{1}{3}e^{-3t} - \frac{2e^{-3t}}{9} - \frac{2e^{-3t}}{27t} + \frac{C}{t^2}$$

	u	dv
$+$	t^3	e^{-3t}
$-$	$3t^2$	$-\frac{1}{3}e^{-3t}$
$+$	$6t$	$\frac{1}{9}e^{-3t}$
$-$	6	$-\frac{1}{27}e^{-3t}$
		$\frac{1}{81}e^{-3t}$

2. Rewrite the Bernoulli equation $t^2y' + 2ty - y^3 = 0$ as a linear equation. (12 points)

$$t^2y' + 2ty = y^3$$

$n = 3$ $\quad (1-n)y^{-n} = (1-3)y^{-3} = -2y^{-3}$

$$-2y^{-3}t^2y' + 2t(-2y^{-3})y = -2y^{-3}y^3$$

$z = y^{-2}$

$z' = -2y^{-3}y'$

$$-2t^2y^{-3}y' - 4ty^{-2} = -2$$

$$t^2z' - 4tz = -2 \quad \text{or}$$

$$z' - \frac{4}{t}z = \frac{-2}{t^2}$$

3. Solve the exact equation $(x+y)^2dx + (2xy + x^2 - 1)dy = 0, y(1) = 1$. (15 points)

$(x^2 + 2xy + y^2)$

$$\int (x+y)^2dx = \tfrac{1}{3}(x+y)^3 + g(y) \quad \text{or} \quad \int x^2 + 2xy + y^2dx = \tfrac{1}{3}x^3 + x^2y + xy^2 + g(y)$$

$$\int 2xy + x^2 - 1\,dy = xy^2 + x^2y - y + h(x)$$

$$\varphi(x,y) = \tfrac{1}{3}x^3 + xy^2 + x^2y - y + K$$

$$\tfrac{1}{3}x^3 + xy^2 + x^2y - y = K$$

$$\tfrac{1}{3}(1)^3 + (1)(1)^2 + (1)^2(1) - (1) = K$$

$$\tfrac{1}{3} + 1 = K \rightarrow K = 4/3$$

$$\varphi(x,y) = \tfrac{1}{3}x^3 + xy^2 + x^2y - y - \tfrac{4}{3}$$

or

$$\tfrac{1}{3}x^3 + xy^2 + x^2y - y = \tfrac{4}{3}$$

4. Suppose a tank of water initially contains 50L of pure water. Salt water, with a concentration of 100g/L is pumped into the tank at a rate of 2L/sec. Suppose that the well-mixed solution is pumped out of the tank at the same rate.

a. Write a differential equation that models this situation. (5 points)

$$\text{Rate in} = \frac{100g}{L}\cdot\frac{2L}{sec} = \frac{200g}{sec} \qquad \text{Rate out} = \frac{A\,g}{50L}\cdot\frac{2L}{sec} = \frac{A}{25}$$

$$\frac{dA}{dt} = 200 - \frac{A}{25}$$

$$A(0)=0$$

b. Solve the differential equation in part a. (10 points)

$$\frac{dA}{dt} = -\frac{1}{25}(A-5000)$$

$$A(t) = 5000 - 5000e^{-1/25\,t}$$

$$\int \frac{dA}{A-5000} = \int -\frac{1}{25}\,dt$$

$$\ln|A-5000| = -\frac{1}{25}t + C$$

$$A-5000 = e^{-1/25 t + C} = A_0e^{-1/25 t}$$

$$A(t) = A_0e^{-1/25 t} + 5000$$

$$0 = A_0e^{0} + 5000 \qquad A_0 = -5000$$

c. What is the equilibrium amount of salt in the tank? How long will it take to achieve 99% of this concentration? You may round your answer here to two decimal places. (5 points)

5000g is equilibrium

$99\% \times 5000 = 4950$

$4950 = 5000 - 5000e^{-1/25 t}$

$0.01 = e^{-1/25 t}$

$-4.6051701 = -\frac{1}{25}t$

$t = 115.129$

≈ 115 seconds

5. Use Euler's method to estimate the solution to the IVP $y' = y(3-ty), y(0) = 0.5$. If you want to know the value of $y(2)$, and will estimate it using 10 steps, find the first three steps of this calculation. (You should use a minimum of 4 decimal places. Use technology to find the rest.) (15 points)

$\Delta t = 0.2 \qquad \frac{2-0}{10} = \frac{1}{5} = 0.2$

$m_0 = 0.5(3 - 0.5(0)) = 1.5$

$y_1 = 1.5(0.2) + 0.5 = 0.8$

$m_1 = 0.8(3 - 0.8(0.2)) = 2.272$

$y_2 = 2.272(0.2) + 0.8 = 1.2544$

$m_2 = 1.2544(3 - 1.2544(0.4)) = 3.13379$

$y_3 = 3.13379(0.2) + 1.2544 = 1.88116$

$y(2) \approx 1.7576$

6. Solve for the general solution for the first order homogeneous equation $\frac{dy}{dx} = \frac{x^2-3y^2}{2xy}$. (15 points)

$y = vx,\ y' = v'x + v \quad v = y/x$

$v'x + v = \frac{x^2 - 3v^2x^2}{2xvx} = \frac{x^2(1-3v^2)}{2x^2v} = \frac{1-3v^2}{2v}$

$v'x = \frac{1-3v^2}{2v} - v\left(\frac{2v}{2v}\right) = \frac{1-3v^2-2v^2}{2v} = \frac{1-5v^2}{2v}$

$\int dv \cdot \frac{2v}{1-5v^2} = \int \frac{1}{x}dx$

$u = 1-5v^2 \quad -1/5\, du = 2v\, dv$

$du = -10v\, dv$

$\int -\frac{1}{5}\frac{du}{u} = \int \frac{1}{x}dx \qquad -\frac{1}{5}\ln u = \ln x + c$

$u^{-1/5} = Ax$

$u = Bx^{-5}$

$1-5v^2 = B/x^5$

$1 - 5(y/x)^2 = B/x^5$

$5\frac{y^2}{x^2} = 1 - B/x^5$

$\boxed{5y^2 = x^2 - B/x^3}$

7. Use improved Euler's method to estimate the first three steps of the calculation for the ODE $y' = 3 + t - y, y(0) = 1$ using the step size $h = 0.05$. (14 points)

$m_{01} = 3+0-1 = 2 \qquad y_{01} = 1 + 1/2\ 0.05 \cdot 2 = 1.05 \qquad y_1 = 1 + 0.05(3 + 0.025 - 1.05)$
$= 1.09875$

$m_{11} = 3 + 0.05 - 1.09875$
$= 1.95125$

$y_{11} = 1.09875 + 1/2(0.05)(1.95125) = 1.147531$

$y_2 = 1.09875 + 0.05(3 + 0.075 - 1.147531) = 1.195123$

$m_{12} = 3 + 0.1 - 1.195123 =$
1.904877

$y_{21} = 1.195123 + 1/2(0.05)(1.904877) = 1.242745$

$y_3 = 1.195123 + 0.05(3 + 0.125 - 1.242745) =$
1.289236

8. Repeat the calculation above using Runge-Kutta for just one step. (12 points)

$k_1 = 3+0-1 = 2 \qquad y_{11} = 1 + 1/2(0.05)(2) = 1.05$

$k_2 = 3 + 0.025 - 1.05 = 1.975 \qquad y_{12} = 1 + 1/2(0.05)(1.975) = 1.049375$

$k_3 = 3 + 0.025 - 1.049375 = 1.975625 \qquad y_{13} = 1 + 0.05(1.9755625) = 1.098781$

$k_4 = 3 + 0.05 - 1.098781 = 1.951219 \qquad y_{14} = 1 + \frac{0.05}{6}(2 + 2(1.975) + 2(1.975625) + 1.95119)$

$= 1.098771 = y_1$

$(0.05,\ 1.098771)$

9. Verify that $x, x^2, \frac{1}{x}$ are solutions to the ODE $x^3y''' + x^2y'' - 2xy' + 2y = 0$. (12 points)

$y = x$ $\quad x^3(0) + x^2(0) - 2x(1) + 2x = 0$ ✓

$y' = 1$

$y'' = y''' = 0$ $\quad x^3(0) + x^2(2) - 2x(2x) + 2(x^2) = 2x^2 - 4x^2 + 2x^2 = 0$ ✓

$y = x^2$ $\quad x^3\left(-\frac{6}{x^4}\right) + x^2\left(\frac{2}{x^3}\right) - 2x\left(-\frac{1}{x^2}\right) + 2\left(\frac{1}{x}\right) =$

$y' = 2x$

$y'' = 2$ $\quad -\frac{6}{x} + \cancel{\frac{2}{x}} + \cancel{\frac{2}{x}} + \frac{2}{x} = 0$ ✓

$y''' = 0$

$y = \frac{1}{x} = x^{-1}$ $\quad$ they are all solutions

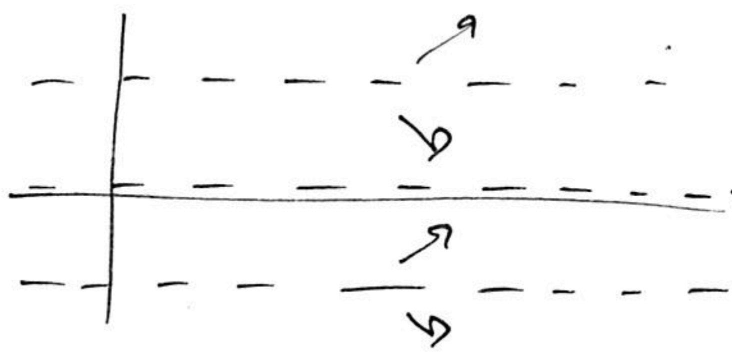
$y' = -x^{-2} = -\frac{1}{x^2}$

$y'' = +2x^{-3} = \frac{2}{x^3}$

$y''' = -6x^{-4} = \frac{-6}{x^4}$

10. Graph the direction field for the equation $y' = y(y^2 - 4)$ by hand. Note any equilibria and describe their stability. (10 points)

$y(y-2)(y+2)$



$y = 2$ unstable (threshold)

$y = 0$ stable

$y = -2$ unstable

11. Use the properties of the Uniqueness and Existence Theorem to determine where the differential equation $y' = \frac{\ln(x^2-1)}{\sqrt{4-x^2-y^2}}$ is guaranteed to have a solution. Sketch the graph of the restrictions. (15 points)

$f = \frac{\ln(x^2-1)}{\sqrt{4-x^2-y^2}}$

f_y does not add new restrictions

$\sqrt{4-x^2-y^2} = 0$

$4 - x^2 - y^2 = 0 \leftarrow$ or $x^2+y^2 > 4$

$x^2 + y^2 = 4$ circle radius 2

$\ln(x^2-1) \rightarrow$ undef.

$x^2 - 1 \leq 0$

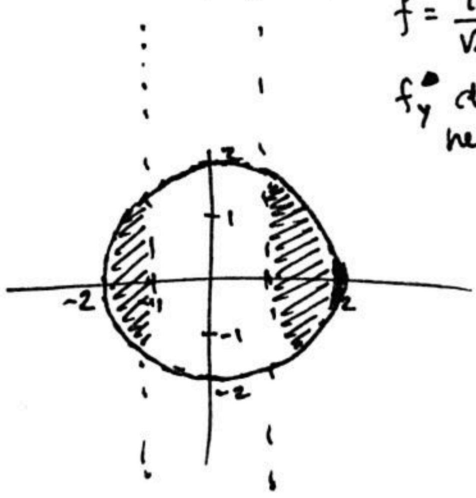
$x^2 \leq 1$

$-1 \leq x \leq 1$

does not exist inside this interval or outside the circle of radius 2

not defined on circle

defined on shaded regions



12. Solve the differential equation $\frac{dy}{dx} = \frac{y^2+1}{xy}, y(1) = 2$ using separation of variables. (15 points)

$$\int \frac{y \cdot dy}{y^2+1} = \int \frac{1}{x}\, dx$$

$$\ln|y^2+1| = \ln x + C$$

$$y^2+1 = Ax$$

$$5 = A$$

$$y^2+1 = 5x \quad \text{or} \quad y = \sqrt{5x-1}$$

13. Classify the differential equations by order, linearity and whether it is ordinary or partial. (9 points each)

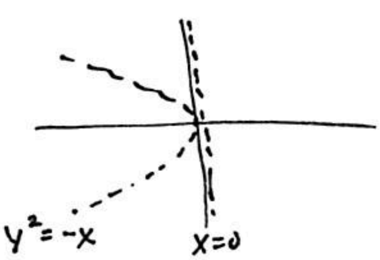
a. $\frac{dy}{dt} = t^2y - \cos t$ 1st order, linear, ordinary

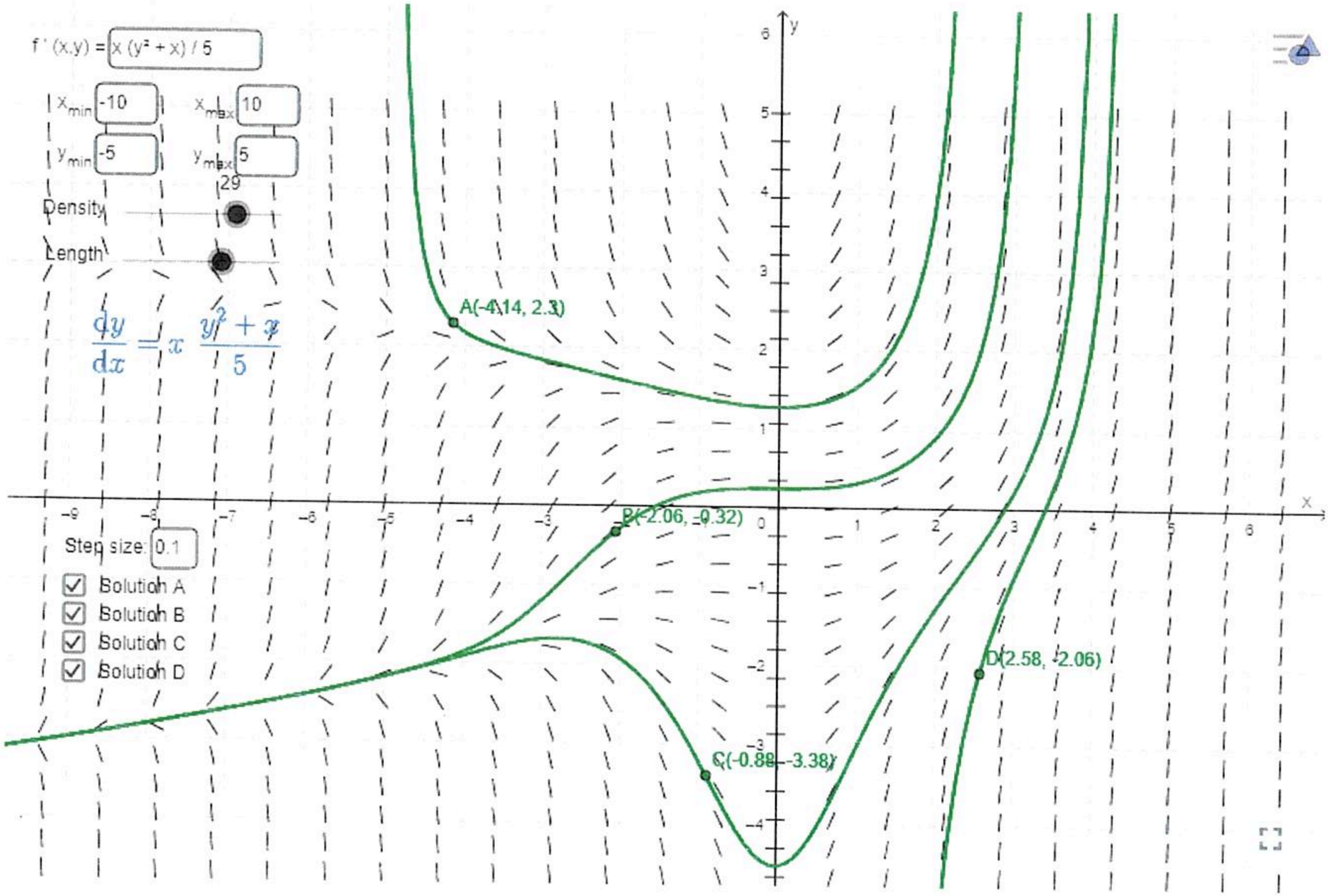
b. $\left(\frac{dy}{dy}\right)^2 = \ln t + y$ 1st order, nonlinear, ordinary

c. $u_{xx} - u_{yy} = u_{xy}$ 2nd order, linear, partial

14. Use technology to graph the direction/slope field for the differential equation $\frac{dy}{dx} = \frac{x(y^2+x)}{5}$. Select two different sets of initial conditions and plot the trajectory of the graph from that point. Note any equilibria (points or lines where the slope is zero). Attach your graphs. (12 points)

null clines at $x = 0$ and $y^2 = -x$





f ' (x,y) = x (y² + x) / 5
x_{min} -10
x_{max} 10
y_{min} -5
y_{max} 5
29
Density
Length
$\frac{dy}{dx} = x\,\frac{y^2+x}{5}$
Step size: 0.1
Solution A
Solution B
Solution C
Solution D
A(-4.14, 2.3)
B(-2.06, -0.32)
C(-0.88, -3.38)
D(2.58, -2.06)
x
y

15. The brewing pot temperature of coffee is 180º F. The room temperature is 76º F. After 5 minutes, the temperature of the coffee is 168º F.

a. Write a differential equation to model this problem. (3 points)

$$\frac{dT}{dt} = k(T-76) \qquad T(0)=180 \qquad T(5)=168$$

b. Solve the equation with the given initial conditions. (5 points)

$$\int \frac{dT}{T-76} = \int k\,dt$$

$$\ln|T-76| = kt + c$$

$$T-76 = e^{kt+c} = T_0 e^{kt}$$

$$T(t) = T_0 e^{kt} + 76$$

$$180 = T_0 e^0 + 76$$

$$104 = T_0$$

$$T(t) = 104 e^{kt} + 76$$

$$168 = 104 e^{5k} + 76$$

$$0.8846 = \frac{92}{104} = e^{5k}$$

$$\ln\left|\frac{23}{26}\right|/5 = k = -0.0245204644$$

c. How long will it take for the coffee to reach a serving temperature of 155º F? (3 points)

$$T(t) = 104 e^{\frac{\ln(23/26)}{5}t} + 76$$

$$\frac{155-76}{104} = e^{\ln(23/26)/5\,t}$$

$$\ln\frac{79}{104} = \ln(23/26)/5 \cdot t$$

$$t = \frac{\ln(79/104)\cdot 5}{\ln(23/26)} = 11.2127 9932$$

approximalty 11 minutes

11 minutes, 13 seconds

Modified Euler': $y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right]$

Runge-Kutta: $y_{n+1} = y_n + h\left(\frac{k_{n1}+2k_{n2}+2k_{n3}+k_{n4}}{6}\right)$,

$$k_{n1} = f(t_n, y_n), k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right),$$

$$k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right), k_{n4} = f(t_n + h, y_n + hk_{n3})$$