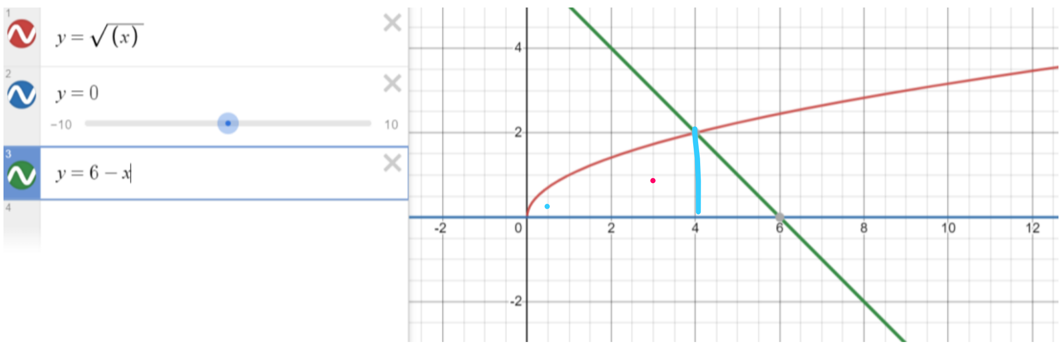
9/6/2022

Physics applications—center of mass (2.6)
Probability – means
Exponentials/logs/growth/decay (2.7/2.8)
Hyperbolic trig functions (2.9)

Centers of Mass of a 2-dimensional lamina (thin sheet)

In the one-variable, with constant density, we are finding the geometric center.



Find the center of mass for lamina bounded by $y = \sqrt{x}, y = 0, y = 6 - x$ with constant density ρ.

First find the total mass:

$$M = \int_a^b \rho[f(x) - g(x)]dx = \rho A$$

The total mass is the area times the density where the integral is the area, and the density is constant.

To find the center of mass in the x-direction, divide the moment of mass from the y-axis by the total mass.

$$\bar{x} = \frac{M_y}{M}$$

$$M_y = \int_a^b \rho x[f(x) - g(x)]dx$$

To find the center of mass in the y-direction, divide the moment of mass from the x-axis by the total mass:

$$\bar{y} = \frac{M_x}{M}$$

$$M_x = \rho \int_a^b \frac{1}{2}[f^2(x) - g^2(x)]dx$$

For this example:

$$M = \rho\left[\int_0^4 \sqrt{x}dx + \int_4^6 6 - xdx\right] = \rho\left\{\frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 + \left[6x - \frac{1}{2}x^2\right]_4^6\right\} = \rho\left[\frac{2}{3}(8) + 36 - 18 - 24 + 8\right] = \frac{22\rho}{3}$$

$$M_y = \rho\left[\int_0^4 x\sqrt{x}dx + \int_4^6 x(6-x)dx\right] = \rho\left[\frac{2}{5}x^{\frac{5}{2}}\Big|_0^4 + \left[3x^2 - \frac{1}{3}x^3\right]_4^6\right] =$$

$$\rho\left[\frac{2}{5}(32) + 108 - 72 - 48 + \frac{64}{3}\right] = \rho\left(\frac{332}{15}\right)$$

$$M_x = \frac{\rho}{2}\left[\int_0^4 \left(\sqrt{x}\right)^2 dx + \int_4^6 (6-x)^2 dx\right] = \frac{\rho}{2}\left[\int_0^4 xdx + \int_4^6 36 - 12x + x^2 dx\right] =$$

$$\frac{\rho}{2}\left[\frac{1}{2}x^2\Big|_0^4 + \left[36x - 6x^2 + \frac{1}{3}x^3\right]_4^6\right] = \frac{\rho}{2}\left[8 + 216 - 216 + 72 - 144 + 96 - \frac{64}{3}\right] = \frac{\rho}{2}\left[\frac{32}{3}\right) = \frac{16}{3}\rho$$

$$\bar{x} = \frac{M_y}{M} = \frac{\rho\left(\frac{332}{15}\right)}{\frac{22\rho}{3}} = \frac{332}{15} \times \frac{3}{22} = \frac{166}{55}\ (\approx 3.02)$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{16}{3}\rho}{\frac{22\rho}{3}} = \frac{16}{3} \times \frac{3}{22} = \frac{8}{11}\ (\approx .73)$$

Center of mass (centroid) is $(\bar{x}, \bar{y}) = \left(\frac{166}{55}, \frac{8}{11}\right)$

Probability density functions: finding the mean of a probability

Suppose that a probability density function is given by $f(x) = kx^2$ on $0 \le x \le 2$. Find k that makes this a probability density function, and then find the mean of the distribution.

$$\int_0^2 kx^2 dx = 1$$

$$\frac{k}{3}x^3\Big|_0^2 = \frac{k}{3}(8) = 1$$

$$k = \frac{3}{8}$$

To find the mean of the density function, $\bar{x} = \int_a^b x\big(f(x)\big)dx\ = \int_0^2 \left(\frac{3}{8}x^2\right)xdx = \int_0^2 \frac{3}{8}x^3 dx = \frac{3}{8}\left(\frac{1}{4}\right)x^4\Big|_0^2$

$$= \frac{3}{8}\left(\frac{1}{4}\right)(16) = \frac{3}{2}$$

2.7 Exponential/Logarithmic antiderivatives

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln(x) + C$$

The $\int \ln(x)\, dx$ will derived in section 3.1 (Integration by Parts).

Remind us of u-substitution and inverse tangent:

$$\int \frac{1}{x-a} dx = \ln|x-a| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C$$

$$\int \frac{x}{1+x^2} dx =$$
$$u = 1 + x^2$$
$$du = 2xdx$$
$$\frac{1}{2} du = xdx$$

$$\int \frac{\left(\frac{1}{2} du\right)}{u} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(1+x^2) + C$$

2.8 Growth and Decay problems
(Exponential Functions)

$$\frac{d}{dx}(a^x) = (\ln a) a^x$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Log is assumed to be log-base 10.
Ln (LN) is assumed to be log-base-e.

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \times \left(\frac{1}{x}\right)$$

$$\ln e^x = x$$
$$a^x = e^{x\ln(a)}$$

$$\log_a x = \frac{\ln(x)}{\ln a}$$

Some of the applications are things like exponential growth.
If you have a population that is growing exponentially at 10% per year and an initial population of 25,000. What is the population have 5 years?
If the problem asks about the rate of growth during year 3, let's say, then the rate is the derivative.

If you have a bank account that is growing exponentially at a rate of 5% per year. What is the accumulated value after 10 years. Accumulation is the area under the curve.

Radioactive decay

Newton's Law of Cooling problem:

The rate of change of the temperature is based on the difference between the room temperature and the object temperature.

2.9 is on Hyperbolic trig functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x)$$

$$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

Inverse hyperbolic trig functions: they exist, and like the “regular” direction functions, they are similar to regular inverse trig function, but with sign changes.

I am never going to ask about inverse hyperbolic trig functions.

This the end of Chapter 2 (application of integration)
The next chapter is on integration techniques