9/29/2022

Sequences (5.1)
Intro to Infinite Series (5.2-?)

Sequences
A sequence is a set of numbers in a given order. An element in a sequence is identified by a subscript.

a_n is the nth element in the sequence. Sometimes the sequence itself is identified by this generic term a_n.

Explicit formula, and an implicit or recursive formula method for determining the values in the sequence.

Explicit formulas:

$$a_n = \frac{n+1}{n^2}$$

$$\left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \dots\right\}$$

$$a_1 = 2, a_2 = \frac{3}{4}, a_{10} = \frac{11}{100}$$

Recursive formula, you need both a formula that depends on other terms in the sequence, and a seed (which is an initial value for a term or terms that are needed to apply the formula the first time).

Fibonacci sequence

$$a_0 = 1, a_1 = 1$$

$$a_n = a_{n-1} + a_{n-2}$$

$$\{1, 1, 2, 3, 5, 8, 13, \dots.\}$$

Two most common series types are arithmetic and geometric.

Arithmetic sequences involve a common difference, which is to say that you add a common constant each time n increases.

$$a_n = a_{n-1} + d$$

$$\{1, 3, 5, 7, 9, 11, \dots\}$$

I can also rewrite this sequence in terms of its initial value, a_0.

$$a_n = (d)n + a_0$$

$$a_n = 2n + 1$$

Geometric sequence involved multiplying a initial value (or the previous value) by a constant multiplier to get from term to term.

$$a_n = a_{n-1}r$$

r is called the common ratio

$$\{1, 2, 4, 8, 16, 32, \dots\}$$

$$a_n = (a_0)r^n$$

$$a_n = (1)(2^n) = 2^n$$

Factorial:

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

$$7! = 7(6)(5)(4)(3)(2)(1) = 5040$$

Sometimes the explicit formulas are written as:

$a_{n+1} = f(a_n)$ instead of $a_n = f(a_{n-1})$

Suppose I want to come up with a formula for the sequence:

$$\left\{\frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \dots\right\}$$

Some common sequences to consider:

$$n: 1, 2, 3, 4, 5, 6, \dots$$

$$n^2: 1, 4, 9, 16, 25, 36, 49, \dots$$

$$n^3: 1, 8, 27, 64, 125, \dots$$

$$n!:$$
$$0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040 \dots$$
$$\{1, 1, 2, 6, 24, 120, 720, 5040, \dots\}$$

$$2^n: 1, 2, 4, 8, 16, 32, \dots$$
$$3^n: 1, 3, 9, 27, 81, 243, \dots$$

Are the numbers I have like these sequences, but shifted.

$$n^2 + 1: 2, 5, 10, 17, 26, \dots$$

$$\left\{\frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \dots\right\}$$

The denominator is powers of 2.
If I started with n=1, then the denominator is just 2^n.
If I started with n=0, then this is just 2^{n+1}

Numerator: something like $2^n - 1$

Starting with n=1

$$a_n = \frac{2^{n+1} - 1}{2^n}$$

Or with n=0

$$a_n = \frac{(2^{n+2} - 1)}{2^{n+1}}$$

$$\left\{\frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \dots\right\}$$

$$\left\{2 - \frac{1}{2}, 2 - \frac{1}{4}, 2 - \frac{1}{8}, 2 - \frac{1}{16}, \dots\right\}$$

$$2 - \frac{1}{2^{n+1}}$$

(for n=0 start)

Sometimes we have alternating signs.

$$\left\{1, -x, \frac{x^2}{2}, -\frac{x^3}{6}, \frac{x^4}{24}, -\frac{x^5}{120}, \dots\right\}$$

$$(-1)^n = 1, -1, 1, -1, 1, -1, \dots$$

$$\cos(n\pi) = 1, -1, 1, -1, \dots$$

$$\left\{\frac{1}{1}, -\frac{x}{1}, \frac{x^2}{2}, -\frac{x^3}{6}, \frac{x^4}{24}, -\frac{x^5}{120}, \dots\right\}$$

$$a_n = \frac{(-1)^n x^n}{n!}$$

The limit of a sequence

Think of $a_n = f(n)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n)$$

What happens to the sequence as n gets very big?

If the sequence goes off to infinity, or fails to converge to a specific single value, then we say the sequence diverges. If the sequence converges (the limit is defined and finite) to a single finite value, then we say the sequence converges.

$$\lim_{n\to\infty} 2 - \frac{1}{2^{n+1}} = 2 - \lim_{n\to\infty} \frac{1}{2^{n+1}} = 2 - 0 = 2$$

Converges to 2

$$\lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{n}{n^2} + \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n^2} = 0 + 0 = 0$$

Converges to 0

$$\lim_{n\to\infty} \sin(n) = ?$$

This limit doesn't exist because while the values are bounded between -1 and 1, it just oscillates and doesn't go anywhere.

Diverges

$$\lim_{n\to\infty} \frac{n!}{2^n} = \infty$$

$$\{1, \frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16}, \frac{120}{32}, \frac{720}{64}, \frac{5040}{128}, \dots\}$$

This sequence diverges because it goes to infinity.

$$\lim_{n\to\infty} \frac{(-1)^n n!}{2^n} = DNE$$

$$\{1, -\frac{1}{2}, \frac{2}{4}, -\frac{6}{8}, \frac{24}{16}, -\frac{120}{32}, -\frac{720}{64}, \frac{5040}{128}, \dots\}$$

Diverge

Squeeze Theorem

$$g(x) \le f(x) \le h(x)$$

$$\lim_{n\to\infty} g(n) \le \lim_{n\to\infty} f(n) \le \lim_{n\to\infty} h(n)$$

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = ?$$

$$-1 \le \sin(n) \le 1$$

$$-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty} -\frac{1}{n} \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le 0$$

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

Bounded and Monotonic

Bounded sequence is a sequence for which we can find a number that is always less (or equal to) than any value of the sequence (bounded below), and we can find a number that is always greater than (or equal to) any value in the sequence (bounded above).

For instance, $a_n = \sin(n)$ is a bounded sequence. Because all values are less than or equal to 1, and all values are greater than or equal to -1.

$$a_n = 2 - \frac{1}{2^{n+1}}$$

This sequence is bounded above by 2. It's bounded below by 0.
Could say that the sequence is bounded above by 3. It's bounded below by 1.

Monotonic/Monotonicity

The sequence is always increasing or always decreasing. (it never changes direction).

Graphing the function that defines the sequence can help you determine this, alternatively, you can take the derivative and determine if the first derivative is always positive or negative.

$$a_n = \frac{1}{n}$$
$$f(n) = \frac{1}{n}$$
$$f(x) = \frac{1}{x}$$
$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$
$$f'(n) = -\frac{1}{n^2}$$

This function is always negative and therefore, always decreasing. Therefore, monotonic.
Upper bound would be the maximum value (has to be 1 at n=1), and bounded below by 0 because $\frac{1}{n}$ is always positive.

One caveat: a function that is monotonic after a finite number of terms, can also be used where "monotonic" are needed.

$$a_n = \frac{2^n}{n!}$$

$$\left\{1, 2, \frac{4}{2} = 2, \frac{8}{6} = \frac{4}{3}, \frac{16}{24} = \frac{2}{3}, \frac{32}{120} = \frac{4}{15}, \dots\right\}$$

Bounded and monotonic theorem:
A sequence that is both bounded (above and below) and is monotonic converges to some limit.

This theorem does not tell us what the limit is, only that it exists.

Series:
A series is the sum of a sequence.

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

$$S_n = \sum_{i=1}^{n} \frac{1}{i}$$

Nth partial sum

$$S_5 = \sum_{i=1}^{5} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

Some infinite sums will diverge (they may go to infinity or may never converge to a specific value). Some will converge to a finite value.

Divergent:

$$\sum_{i=0}^{\infty} (-1)^i$$

$$1 + (-1) + 1 + (-1) + \cdots$$

$$S_n = \{1,0,1,0,1,0,1,0, \dots\}$$

The limit of the sequence of partial sums does not exist.

Convergent:

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

$$S_n = \{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \dots\}$$

Next time we'll pick up with geometric and telescoping series, and then continue with series tests.