9/20/2022

Integration by Tables (3.5)
Numerical Integration (3.6)

Integration by Tables:
The table use integration techniques that we learned with generic coefficients to establish formulas for integrals or at least reduction formulas. The trick is in picking the right formula, and in making any appropriate substitutions to match the correct formula.

We will use the table of integrals posted in the course.

Example.

$$\int \frac{x}{\sqrt{1+2x}}dx$$

1) Look through the integral table to find a formula that best fits this problem type.
2) If none match exactly, consider a substitution
3) Identify any coefficients, and pull out constants as necessary.

(3.7) $$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2x}{a}\sqrt{ax+b} - \frac{4b}{3a^2}(ax+b)^{3/2} + C$$

$$a = 2, b = 1$$

$$\int \frac{x}{\sqrt{1+2x}}dx = \frac{2x}{2}\sqrt{1+2x} - \frac{4(1)}{3(2)^2}(1+2x)^{\frac{3}{2}} + C = x\sqrt{1+2x} + \frac{1}{3}(1+2x)^{\frac{3}{2}} + C$$

Example.

$$\int \frac{x}{1+e^{-x^2}}dx$$

(8.10) $$\int \frac{1}{a+be^{kx}}dx = \frac{1}{ak}\left[kx - \ln\left|a+be^{kx}\right|\right] + C$$

Let $u = -x^2, du = -2xdx, -\frac{1}{2}du = xdx$

$$\int -\frac{1}{2}\left(\frac{1}{1+e^u}\right)du = -\frac{1}{2}\left[\frac{1}{(1)(1)}\{(1)u - \ln|1+e^u|\}\right] + C = -\frac{1}{2}\left[-x^2 - \ln\left|1+e^{-x^2}\right|\right] + C$$

$$a = 1, b = 1, k = 1$$

Example.

$$\int \tan^2\left(\frac{x}{2}\right)dx$$

(4.34) $$\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x + C$$

$$a = \frac{1}{2}$$

$$\int \tan^2\left(\frac{x}{2}\right)dx = \frac{1}{\frac{1}{2}}\tan\left(\frac{1}{2}x\right) - x + C = 2\tan\left(\frac{x}{2}\right) - x + C$$

Example.

$$\int \frac{\cos(x)}{\sin^2 x + 2\sin x}dx$$

Let $u = \sin x\,, du = \cos x\,dx$

$$\int \frac{1}{u^2 + 2u}du = \int \frac{1}{u(u+2)}du$$

(2.17) $$\int \frac{1}{x(ax+b)}dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$$

$$a = 1, b = 2$$

$$\int \frac{1}{u(u+2)}du = \frac{1}{2}\ln\left|\frac{u}{u+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sin(x)}{\sin x + 2}\right| + C$$

Numerical Integration
Only works on definite integral

For example, we cannot integrate functions like $e^{x^2}, \frac{1}{1+x^4}, \sqrt{1+x^3}$,…

Trapezoidal Rule

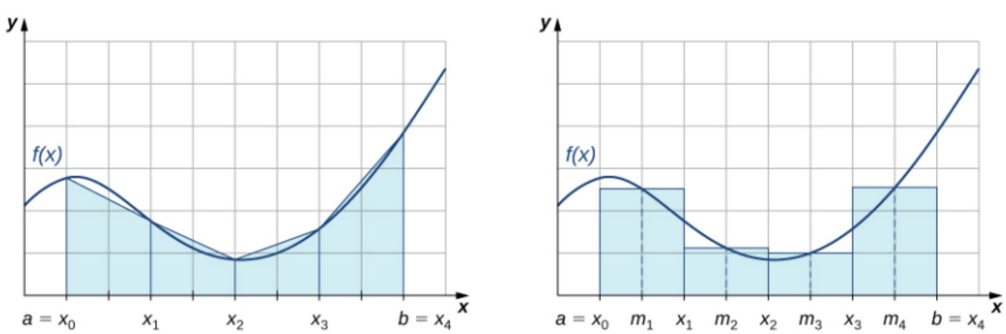


**Figure 3.15** The trapezoidal rule tends to be less accurate than the midpoint rule.

$$A_{trapezoid} = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right)\Delta x$$

$$\frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x + \frac{1}{2}\left(f(x_2) + f(x_3)\right)\Delta x + \frac{1}{2}\left(f(x_3) + f(x_4)\right)\Delta x$$

$$\frac{1}{2}\Delta x[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_3) + f(x_4)] =$$

$$\frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

In general,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$
$$= \frac{(\mathrm{b} - \mathrm{a})}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$\Delta x = \frac{b - a}{n}$$

Example.

$$\int_1^4 \frac{1}{x}dx$$

Use the trapezoidal rule with n=6

$$x_0 = 1, x_n = x_6 = 4, \Delta x = \frac{4 - 1}{6} = \frac{1}{2}$$
$$x_0 = 1, x_1 = 1 + \frac{1}{2} = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3, x_5 = 3.5, x_6 = 4$$

$$\int_1^4 \frac{1}{x}dx \approx \frac{3}{12}\left[\frac{1}{1} + 2\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2.5}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3.5}\right) + \frac{1}{4}\right] = 1.405357 \ldots$$

True value: $ln(4) - ln(1) = ln(4) = 1.386294 \ldots$

Carry enough decimal places (all of them) until the very last step.

Error (absolute value of error) in Trapezoidal Rule

$$E \leq \frac{\max|f''(x)|\,(b - a)^3}{12n^2}$$

$$f(x) = \frac{1}{x} = x^{-1}$$
$$f'(x) = -\frac{1}{x^2} = -x^{-2}$$
$$f''(x) = \frac{2}{x^3} = 2x^{-3}$$

Will be a maximum on the interval at x=1.

$$E \leq \frac{\frac{2}{1^3}(4-1)^3}{12(6)^2} = 0.125$$

If we want to estimate n:

$$n^2 \geq \frac{\max|f''(x)|\,(b-a)^3}{12(E)}$$

We want to estimate our integral to within 0.01 of the true value. How many n are required to ensure that?

$$n^2 \geq \frac{\frac{2}{1}(3)^3}{12(0.01)} = 450$$

$$n \geq 21.21\ldots$$

Always round up. $n = 22$ to guarantee that the error is lower than required.

Simpson's Rule

Approximates the curve as a quadratic polynomial

$$\int_a^b f(x)\,dx \approx$$

$$\frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

The number of intervals for Simpson's Rule must be even.

$$\int_1^4 \frac{1}{x} dx$$

$$x_0 = 1, x_n = x_6 = 4, \Delta x = \frac{4-1}{6} = \frac{1}{2}$$

$$x_0 = 1, x_1 = 1 + \frac{1}{2} = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3, x_5 = 3.5, x_6 = 4$$

$$\int_1^4 \frac{1}{x} dx \approx \frac{3}{18}\left[\frac{1}{1} + 4\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2.5}\right) + 2\left(\frac{1}{3}\right) + 4\left(\frac{1}{3.5}\right) + \frac{1}{4}\right] = 1.38769\ldots$$

True value: $ln(4) - ln(1) = ln(4) = 1.386294\ldots$

Error Rule for Simpson's Rule:

$$E \leq \frac{\max|f^{IV}(x)|\,(b-a)^5}{180n^4}$$

$$f''(x) = \frac{2}{x^3} = 2x^{-3}$$

$$f'''(x) = -6x^{-4} = -\frac{6}{x^4}$$

$$f^{IV}(x) = \frac{24}{x^5} = 24x^{-5}$$

$$E \leq \frac{\frac{24}{1^5}(4-1)^5}{180(6)^4} = 0.025$$

Rearrange to find n. Not only do you have to round up, but you have to round up to the next EVEN number.

$$n^4 \geq \frac{\frac{24}{1}(3)^5}{180(0.01)} = 3240$$

$$n \geq 7.5 \ldots$$

Round to 8.

Next time, Improper Integrals.