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Partial Fractions (3.4)
Overview of Integration Techniques

General Rational functions: integration methods to try before using partial fractions.

1) Is the numerator the same degree or higher than the denominator: if the answer is yes, you need to do long division to simplify the expression. The terms that divide out are likely to just be power rule terms. The remainder will be a lower degree than the denominator, and then you may consider partial fractions.

2) Can the denominator be factored?
If the denominator is linear, then just go to log rules
If the denominator is quadratic but can't be factored, then consider inverse tangent functions, and you may need to complete the square.
If the denominator is cubic or larger, then you will need to factor. You may want to graph the function, either in your calculator or Desmos to find all the rational zeros. Then use long division to factor.
If you can't find any rational zeros to help you factor the denominator with division, then check that you've copied the denominator correctly (signs, coefficients, the correct problem, etc.)
Review factoring of sum/difference of cubes, and factoring by grouping!

3) In order to apply partial fractions: the denominator must be factored as much as possible (linear factors, or quadratic factors that can't be factored further (complex roots)), and the numerator must be a lower degree than the denominator.

Example of linear factors (with no repetition)

$$\int \frac{3x+2}{x^3-x^2-2x}dx = \int \frac{3x+2}{x(x^2-x-2)}dx = \int \frac{3x+2}{x(x-2)(x+1)}dx$$

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$\frac{A}{x}\frac{(x-2)(x+1)}{(x-2)(x+1)} + \frac{B}{x-2}\frac{x(x+1)}{x(x+1)} + \frac{C}{x+1}\frac{x(x-2)}{x(x-2)} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}$$

$$A(x-2)(x+1) + Bx(x+1) + Cx(x-2) = 3x+2$$

Method 1: Algebra way

$$A(x^2-x-2) + Bx^2 + Bx + Cx^2 - 2Cx =$$

$$Ax^2 - Ax - 2A + Bx^2 + Bx + Cx^2 - 2Cx = 0x^2 + 3x + 2$$

$$(Ax^2 + Bx^2 + Cx^2) + (-Ax + Bx - 2Cx) + (-2A) = 0x^2 + 3x + 2$$
$$x^2(A+B+C) + x(-A+B-2C) + (-2A) = 0x^2 + 3x + 2$$

$$x^2: A + B + C = 0$$
$$x: -A + B - 2C = 3$$
$$1: -2A = 2$$

$$-2A = 2 \rightarrow A = -1$$

$$-1 + B + C = 0 \rightarrow B + C = 1$$
$$-(-1) + B - 2C = 3 \rightarrow B - 2C = 2$$

$$B + C = 1$$
$$-B + 2C = -2$$

$$3C = -1$$
$$C = -\frac{1}{3}$$
$$B + \left(-\frac{1}{3}\right) = 1$$
$$B = \frac{4}{3}$$

$$A = -1, B = \frac{4}{3}, C = -\frac{1}{3}$$

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{-1}{x} + \frac{\left(\frac{4}{3}\right)}{x-2} + \frac{\left(-\frac{1}{3}\right)}{x+1}$$

$$\int \frac{3x+2}{x^3 - x^2 - 2x} dx = \int \frac{3x+2}{x(x^2 - x - 2)} dx = \int \frac{3x+2}{x(x-2)(x+1)} dx =$$

$$\int \frac{-1}{x} + \frac{\left(\frac{4}{3}\right)}{x-2} + \frac{\left(-\frac{1}{3}\right)}{x+1} dx = -\int \frac{1}{x} dx + \frac{4}{3}\int \frac{1}{x-2} dx - \frac{1}{3}\int \frac{1}{x+1} dx =$$

$$-\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C$$
$$\ln\left|\frac{(x-2)^{\frac{4}{3}}}{(x-1)^{\frac{1}{3}}x}\right| + C$$

Method 2: Short-way (zeros method).

$$A(x-2)(x+1) + Bx(x+1) + Cx(x-2) = 3x + 2$$

Reduce the problem by picking values of x that will make some of the coefficients disappear (when the factors are zero).

$$x = 2$$
$$A(0)(2+1) + B(2)(2+1) + C(2)(0) = 3(2) + 2$$

$$6B = 8$$
$$B = \frac{4}{3}$$

$$x = -1$$
$$A(-1-2)(0) + B(-1)(0) + C(-1)(-1-2) = 3(-1) + 2$$
$$3C = -1$$
$$C = -\frac{1}{3}$$

$$x = 0$$
$$A(0-2)(0+1) + B(0)(0+1) + C(0)(0-2) = 3(0) + 2$$
$$-2A = 2$$
$$A = -1$$

Finish integrating as before using these coefficients.
If you have factors that cannot be zero, but you'll have to select less friendly values of x (that doesn't make things go to zero), and it may only reduce the number of coefficients to solve for. It won't necessarily reduce to just one at a time.

It may help to replace coefficients you can find into the problem to further reduce what needs to be solved for.

Example.

$$\int \frac{2x-3}{x^3+x}dx = \int \frac{2x-3}{x(x^2+1)}dx$$

$$\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

For roots that are not repeated, the numerator should be one degree less than the denominator.

$$\frac{A}{x}\frac{x^2+1}{x^2+1} + \frac{Bx+C}{x^2+1}\left(\frac{x}{x}\right) =$$

$$\frac{2x-3}{x(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x)}{x(x^2+1)}$$

$$0x^2 + 2x - 3 = Ax^2 + A + Bx^2 + Cx$$

$$x^2\!: A + B = 0$$
$$x\!: C = 2$$
$$1\!: A = -3$$

$$B = 3$$

$$\int \frac{2x-3}{x(x^2+1)}dx = \int \frac{-3}{x} + \frac{3x+2}{x^2+1}dx = \int -\frac{3}{x} + \frac{3x}{x^2+1} + \frac{2}{x^2+1}dx$$

$$= -3\ln|x| + \frac{3}{2}\ln|x^2+1| + 2\arctan(x) + C$$

Zeros method for coefficients:

$$A(x^2+1) + (Bx+C)(x) = 2x-3$$
$$x = 0$$
$$A(0+1) + (B(0)+C)(0) = 2(0)-3$$
$$A = -3$$

You can try other values to reduce the FOILing necessary: maybe x=1, x=-1

$$x = 1$$
$$A(2) + (B(1)+C)(1) = 2(1)-3$$
$$2A + B + C = -1$$
$$2(-3) + B + C = -1$$
$$B + C = 5$$

$$x = -1$$
$$2A + (B(-1)+C)(-1) = 2(-1)-3$$
$$2(-3) + B - C = -5$$
$$B - C = 1$$

$$B + C = 5$$
$$B - C = 1$$
$$2B = 6$$
$$B = 3, C = 2$$

What if the factors are repeated?
The numerators will look like what they would for the singular factor, but you will need separate expressions for each repetition…

Suppose you have a denominator like $\frac{1}{(x-1)^3(x^2+4)^2}$

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Example.

$$\int \frac{x-2}{(2x-1)^2(x-1)}\,dx = \int \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}\,dx$$

$$\frac{A}{2x-1}\frac{(2x-1)(x-1)}{(2x-1)(x-1)} + \frac{B}{(2x-1)^2}\frac{x-1}{x-1} + \frac{C}{x-1}\frac{(2x-1)^2}{(2x-1)^2} = \frac{x-2}{(2x-1)^2(x-1)}$$

$$A(2x-1)(x-1) + B(x-1) + C(2x-1)^2 = x-2$$

$$x = 1$$
$$A(2(1)-1)(0) + B(0) + C(2(1)-1)^2 = (1)-2$$

$$C = -1$$

$$x = \frac{1}{2}$$
$$A(0)\left(\frac{1}{2} - 1\right) + B\left(\frac{1}{2} - 1\right) + C(0)^2 = \frac{1}{2} - 2$$
$$-\frac{1}{2}B = -\frac{3}{2}$$
$$B = 3$$

$$x = 0$$
$$A(2(0) - 1)(0 - 1) + B(0 - 1) + C(2(0) - 1)^2 = 0 - 2$$
$$A - B + C = -2$$
$$A - 3 + (-1) = -2$$
$$A - 4 = -2$$
$$A = 2$$

$$\int \frac{2}{2x - 1} + \frac{3}{(2x - 1)^2} + \frac{-1}{x - 1} dx = \ln|2x - 1| - \frac{3}{2}\left(\frac{1}{2x - 1}\right) - \ln|x - 1| + C$$

Set-up problems:

$$\frac{x^3 + x - 1}{x(x - 4)^2(x^2 + 4)(x^2 + 1)^2}$$

$$\frac{A}{x} + \frac{B}{x - 4} + \frac{C}{(x - 4)^2} + \frac{Dx + E}{x^2 + 4} + \frac{Fx + G}{x^2 + 1} + \frac{Hx + I}{(x^2 + 1)^2}$$

Think about strategies overall.