09/01/2022

Work (2.5)
https://tutorial.math.lamar.edu/Classes/CalcII/CalcII.aspx/

In classical, algebra-based physics, we learn that work is force times distance.

$$W = Fd$$

In calculus-based physics

$$W = \int_a^b F(x)dx$$

Or

$$W = \int_a^b d(x)dF$$

Simplest case is probably the spring problem.

Example. Suppose that 5 N stretches a spring 10 cm from equilibrium. Find the work done stretching the spring an additional 5 cm.

$$F = kx$$
$$5 = k(0.1)$$
$$k = 50$$

$$W = \int_{0.1}^{0.15} 50xdx = 25x^2|_{0.1}^{0.15} = 25(0.15^2 - 0.1^2) = 0.3125\ Nm$$

Example. Suppose that a 10 lbs mass stretches a spring 4 inches. Find the work done stretching the spring from equilibrium to an additional 8 inches in length.

$$F = kx$$
$$10 = k\left(\frac{1}{3}\right)$$
$$k = 30$$

$$W = \int_0^{\frac{2}{3}} 30xdx = 15x^2|_0^{\frac{2}{3}} = 15\left(\left(\frac{2}{3}\right)^2\right) = \frac{20}{3} foot - pounds$$

Watch out for units (foot-pounds vs. inch-pounds, or mile-tons, etc. or kilogram masses vs. Newtons or pounds… use F=ma to find Newtons from kilograms, or pounds from slugs).

The distance is from the natural length/equilibrium of the spring. If the length of the spring is provided, you may need to subtract it out to find x.

Generally speaking, you need to find spring constant k from information in the problem.

Problems that are similar:

Gravity problems: $F = \frac{Gm_1m_2}{x^2} = \frac{k}{x^2}$.

Usually have to keep in mind the radius of the body. If you are on Earth, $r \approx 4000\ mi$.

Similar to the gravity is charged particles: $F = \frac{k}{x^2}$.

Variable force problems:

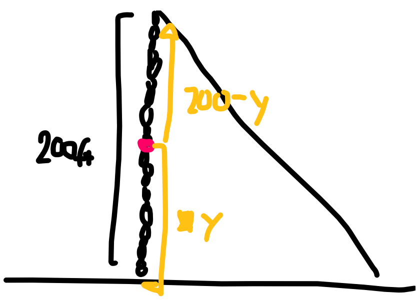
Chains and tank problems (pumping liquid out of a tank)

Chain:

Suppose you have a chain hanging from a crane. It hangs 200 feet from the top. Suppose that it weighs 8 pounds per foot. Calculate the work done in winding up the chain all the way to the top.

Calculate the force:

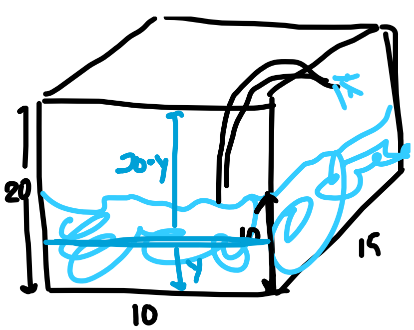
$$F = density \times length$$
$$dF = 8\frac{lbs}{ft} \times dy\ ft = 8\ dy\ (lbs)$$



$$W = \int_0^{200} (200 - y)\,8dy = 8\left[200y - \frac{1}{2}y^2\right]_0^{200} = 8[40{,}000 - 20{,}000] = 160{,}000\ foot - pounds$$

Water tank:

Suppose we have a rectangular tank with a base that is 10 feet by 15 feet that is 20 feet deep. The tank is half-filled with water, with density 62.4 pounds/cubic-foot. Find the work to pump the water out over the top of the tank.

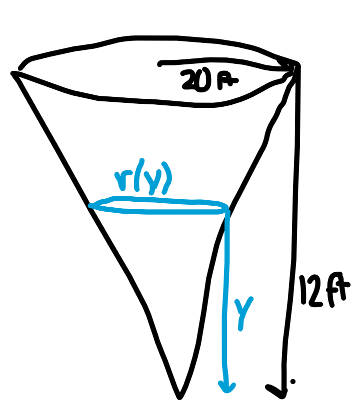


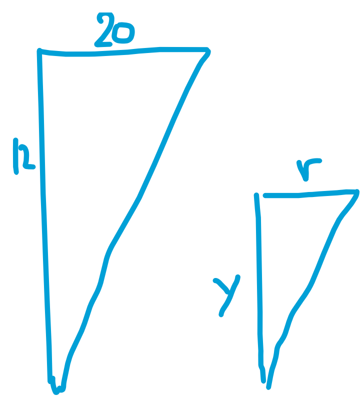
$$F = density \times volume$$
$$F = density \times (area \times dy)$$
$$F = 62.4(10)(15)dy$$

$$W = \int_0^{10} 62.4(10)(15)(20-y)dy = 62.4(150)\left[20y - \frac{1}{2}y^2\right]_0^{10} = 62.4(150)(200-50)$$
$$= 62.4(150)^2 = 140{,}400\ foot - pounds$$

In SI units, the density of water is 1000 N/m^3.

Example.
Suppose we have a conical tank that is 12 feet high with a radius of 20 feet. The tank is full of water. Calculate the work needed to pump all the water out of the tank.





$$\frac{20}{12} = \frac{r}{y} \rightarrow \frac{5}{3}y = r$$

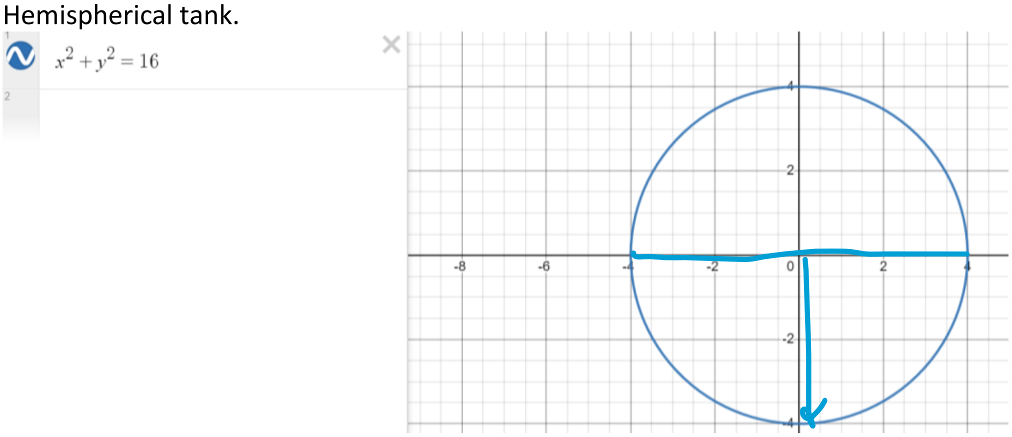
$$F = density \times volume$$

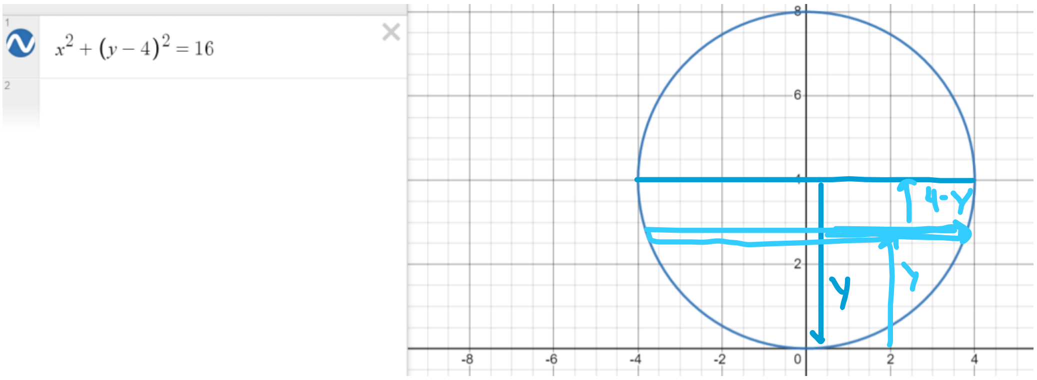
$$F = density \times (area)dy = density \times (\pi r^2)dy = 62.4\left(\pi\left(\frac{5}{3}y\right)^2\right)dy$$

$$W = \int_0^{12} 62.4\pi\left(\frac{25}{9}\right)y^2(12-y)dy = 62.4\left(\frac{25}{9}\right)\pi\int_0^{12} 12y^2 - y^3 dy =$$

$$62.4\left(\frac{25}{9}\right)\pi\left[4y^3 - \frac{1}{4}y^4\right]_0^{12} = 62.4\left(\frac{25}{9}\right)\pi(6912 - 5184) = 299{,}520\pi$$

Hemispherical tank.





Suppose we have a hemispherical tank that has a radius of 4 feet. The tank is full. We want to pump all the water out of the tank over the top. Calculate the work done in doing so.

For a given height y, the radius at that point is x. So, solve the equation (of the circle) for x.

$$x^2 + y^2 - 8y + 16 = 16$$
$$x^2 + y^2 - 8y = 0$$
$$x^2 = 8y - y^2$$
$$r = x = \sqrt{8y - y^2}$$

$$F = density \times volume = density \times (area)dy$$
$$= density \times (\pi r^2)dy = 62.4\left(\pi\left(\sqrt{8y - y^2}\right)^2\right)dy = 62.4\,\pi(8y - y^2)dy$$

$$W = \int_0^4 62.4\,\pi(8y - y^2)(4 - y)dy$$

Ugh! I forgot to record!
I'll post the video and notes from my summer class in the recordings link page.