11/3/2022

Differential Equations (chapter 4)
Basics
Directions Fields/Slope Fields
Numerical Methods

Next Tuesday is Election Day… there is no class.

Differential equation is an equation that contains: a variable, a function, and the function's derivatives.

Simplest differential equation: $y'' = 0$
Complex differential equations are possible: $(x+1)y'' + \frac{3x}{x-4}y' - x^3 y = e^x$

$$y'' - y = 0$$

Classify our differential equations:
Ordinary differential equations vs. partial differential equations
(are functions of only one variable) vs. (the functions are functions of 2 or more variables)

$$y(x)\ vs.\ z(x,y)$$

Typical ordinary differential equation can use prime notation ($y', y'', y''', etc.$), or Leibniz notation $\frac{dy}{dx}, \frac{dy}{dt}, \frac{d^2y}{dx^2}, etc.$ Note the use of d's and primes.

In partial differential equations we use partial derivative notation. They use subscripts instead of primes to indicate which variable the derivative is with respect to. $f_x, f_y, f_{xy}, f_{xxx}, etc.$ Or Leibniz notation use the "del" symbol instead of d. $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, etc.$

$$y'' - y' + 6y = e^x$$

This is an ordinary differential equation.

$$u_{xx} - u_{yy} = 0$$

This is a partial differential equation.

$$\frac{dy}{dx} = y - x$$

This is ordinary.

$$\frac{\partial z}{\partial x} = z + y - x$$

This is partial.

Linearity.
A differential equation is considered linear if the function variables are linear in the differential equation. They are not raised to powers, they are not inside other functions, they are not multiplying each other. What the naked variable terms are doing does not matter.

$$y' - y = e^x$$
Is linear.

$$e^x y'' + x^2 y = \sin(x)$$
Is also linear.

$$y''y = \sin(x)$$
$$y^2 = \frac{y'}{x}$$
$$y'' - y = \sin(y)$$
These are not linear.

Order.
The highest derivative in the equation.

$$y' - e^x y = x$$
This is first order. Because the highest derivative is the first derivative.

$$y'' - y = 0$$
This is second order, because the highest derivative is the second derivative.

We are going to work with all first order problems. Ordinary only. The linearity turns out not to be important for the specific technique that we will look at.

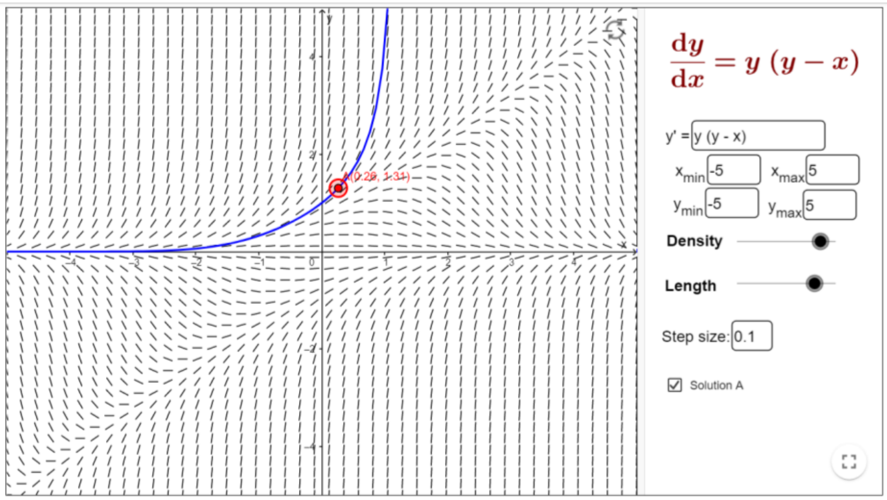
Direction fields/slope fields, in the context of first order differential equations.

Gives us a way to graphically represent the differential equation without having to solve the equation.

$$\frac{dy}{dx} = y(y - x)$$
Create a graph (x on the horizontal axis, and y on the vertical axis), and at each point in the plain (representing an initial condition), we plot a slope from using the differential equation.

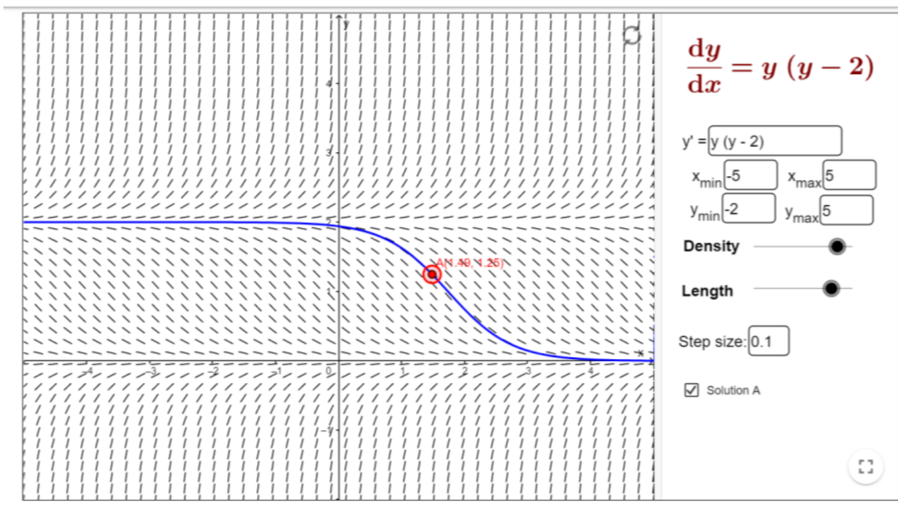
Suppose I'm at the point (1,0). $\frac{dy}{dx} = (0)(0 - 1) = 0$
If I'm at the point $(0,1)$, $\frac{dy}{dx} = (1)(1 - 0) = 1$
Fill in the entire plane with little slopes.

https://www.geogebra.org/m/Pd4Hn4BR



Automonous equations depend only on the function and its derivative, not the independent variable.

$$\frac{dy}{dx} = y(y-2)$$



The slope field is the same going across (because x doesn't change the slope).

Equilibrium is a place where the derivative is zero and the y value doesn't change (because the slope is zero).

Can be stable or unstable. Stable equilibria attract the curves. (0 in the graph above). Or unstable: curves move away from the value (2 in the graph above).

We can have semistable equilibria: move away on one side and move toward on the other. But this requires even powers in the differential equation (repeated roots) : $y(y-2)^2$ would make y=2 semistable.

Euler's method

We pick a point to start. We calculate the slope at the point using the differential equation. Then move along that slope for a small period. Then we recalculate the position. Repeat the whole.

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x$$

$$x_{n+1} = x_n + \Delta x$$

$$\frac{dy}{dx} = f(x, y)$$

Example.

$$\frac{dy}{dx} = y(y - x)$$
$$m_n = f(x, y) = y(y - x)$$

Starting at $y(1) = 2$

$$x_0 = 1, y_0 = 2$$

Step size of 0.1=Δx

$$m_0 = y_0(y_0 - x_0) = (2)(2 - 1) = 2$$
$$y_1 = y_0 + 2\Delta x = 2 + 2(0.1) = 2.2$$
$$x_1 = 1 + 0.1 = 1.1$$

Repeat the process with starting at $(1.1, 2.2)$
If we wanted to estimate $y(2)$ that's 10 steps of 0.1 to get there, so we'd need 10 steps of Euler's to get y_{10} to estimate that point.

Next time we'll continue with more Euler's (in Excel?)
Separable solution methods.