11/29/2022

Calculus with Polar Coordinates
Tangent lines (https://tutorial.math.lamar.edu/classes/calcii/polartangents.aspx)
Arc Length and Area (7.4)

Slope of the tangent line in polar coordinates.

The formula is taken from the idea of a slope of the tangent line in parametric coordinates:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Where t is the parameter. And for this application, we will treat θ as our parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

We also know that $x = r\cos\theta$, $y = r\sin\theta$ and r is a function of θ. i.e. $r(\theta)$

To find $\frac{dy}{d\theta}$, think of $y = r(\theta)\sin\theta$. To get this derivative we need to apply the product rule and differentiate implicitly.
Similarly for $\frac{dx}{d\theta}$.

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\,sin\theta}$$

Example.
Find the slope of the tangent line on the function $r = 3 + 8\sin\theta$, $\theta = \frac{\pi}{6}$.

$$r' = 8\cos\theta$$

$$\frac{dy}{dx} = \frac{(8\cos\theta)\sin\theta + (3 + 8\sin\theta)\cos\theta}{(8\cos\theta)\cos\theta - (3 + 8\sin\theta)\,sin\theta} = \frac{8\sin\theta\cos\theta + 3\cos\theta + 8\sin\theta\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta} =$$

$$\frac{3\cos\theta + 16\sin\theta\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

$$\frac{dy}{dx}\left[\theta = \frac{\pi}{6}\right] = \frac{3\left(\frac{\sqrt{3}}{2}\right) + 16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{8\left(\frac{\sqrt{3}}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 8\left(\frac{1}{2}\right)^2} = \frac{\left(\frac{11\sqrt{3}}{2}\right)}{6 - \frac{3}{2} - 2} = \frac{\left(\frac{11\sqrt{3}}{2}\right)}{\left(\frac{5}{2}\right)} = \frac{11\sqrt{3}}{5}$$

If we wanted to continue to find the equation of the tangent line:

$$r\left(\frac{\pi}{6}\right) = 3 + 8\left(\frac{1}{2}\right) = 3 + 4 = 7$$

$$\left(7, \frac{\pi}{6}\right)$$

$$x = r\cos\theta\,, y = r\sin\theta$$

$$\left(7\left(\frac{\sqrt{3}}{2}\right), 7\left(\frac{1}{2}\right)\right) = \left(\frac{7\sqrt{3}}{2}, \frac{7}{2}\right)$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{7}{2} = \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

Arc length
In parametric form:

$$s = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$x = r\cos\theta\,, y = r\sin\theta$ with θ as the parameter, and $r(\theta)$
In polar form, a and b are angles.

$$s = \int_a^b \sqrt{(r'\sin\theta + r\cos\theta)^2 + (r'\cos\theta - r\sin\theta)^2} d\theta =$$

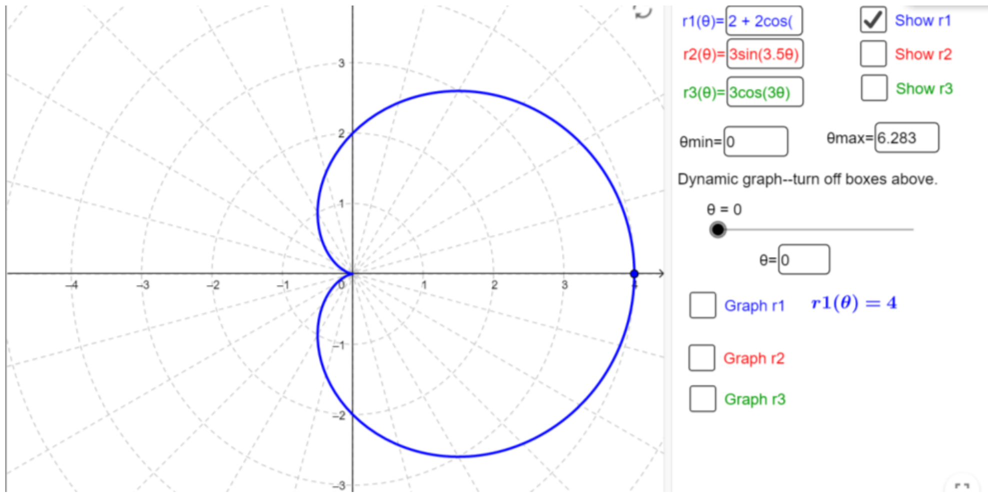
$$\int_a^b \sqrt{(r')^2\sin^2\theta + r'r\sin\theta\cos\theta + r^2\cos^2\theta + (r')^2\cos^2\theta - r'r\cos\theta\sin\theta + r^2\sin^2\theta}\, d\theta =$$

$$\int_a^b \sqrt{(r')^2\sin^2\theta + (r')^2\cos^2\theta + r^2\cos^2\theta + r^2\sin^2\theta}\, d\theta =$$

$$\int_a^b \sqrt{(r')^2(\sin^2\theta + \cos^2\theta) + r^2(\cos^2\theta + \sin^2\theta)} d\theta =$$

$$\int_a^b \sqrt{(r')^2 + r^2} d\theta$$

Example.
Find the length of arc of $r = 2 + 2\cos\theta$ (cardioid)



Use symmetry when possible.

$$r' = -2\sin\theta$$

$$s = \int_a^b \sqrt{(r')^2 + r^2} d\theta = 2\int_0^\pi \sqrt{(-2\sin\theta)^2 + (2 + 2\cos\theta)^2} d\theta =$$

$$2\int_0^\pi \sqrt{4\sin^2\theta + 4 + 8\cos\theta + 4\cos^2\theta}\, d\theta =$$

$$2\int_0^\pi \sqrt{4 + 4 + 8\cos\theta}\, d\theta = 2\int_0^\pi \sqrt{8 + 8\cos\theta}\, d\theta = 2\sqrt{8}\int_0^\pi \sqrt{1 + 1\cos\theta}\, d\theta =$$

$$1 + \cos\theta = 2\sin^2\frac{\theta}{2}$$

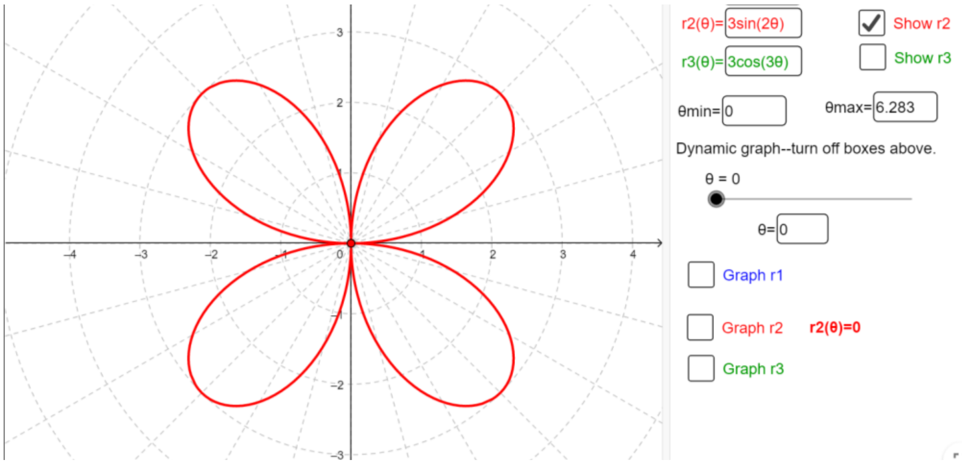
$$2\sqrt{8}\int_0^\pi \sqrt{2\sin^2\frac{\theta}{2}} d\theta = 2\sqrt{8}\sqrt{2}\int_0^\pi \sqrt{\sin^2\frac{\theta}{2}} d\theta = 2\sqrt{16}\int_0^\pi \left|sin\frac{\theta}{2}\right| d\theta =$$

$$8\left(-\cos\frac{\theta}{2}\right)(2)\Big|_0^\pi = 16(-0) - 16(-1) = 16$$

Area in polar graphs.

$$A = \frac{1}{2}\int_a^b [r(\theta)]^2 d\theta$$

Example.
Find the area of one petal of the graph $r(\theta) = 3\sin 2\theta$.



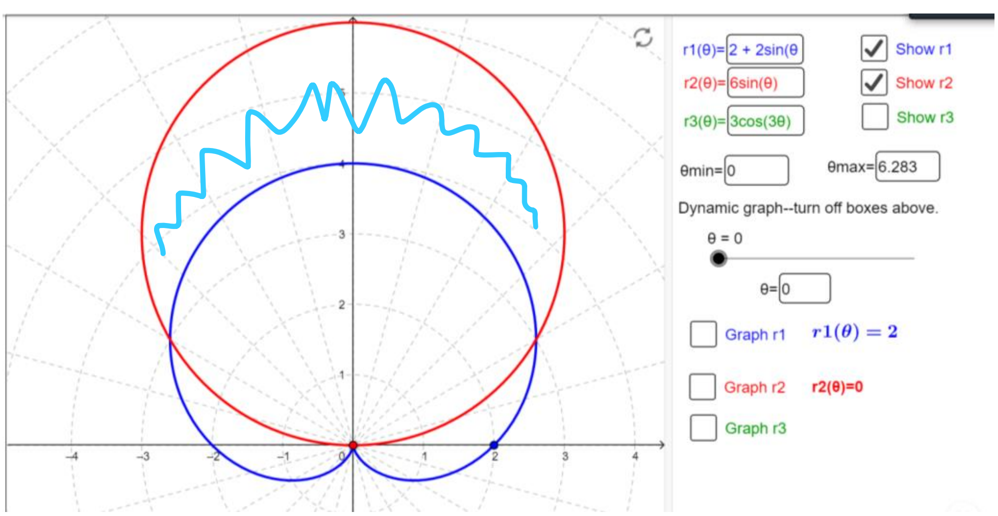
$$3 \sin 2\theta = 0$$
$$\sin 2\theta = 0$$
$$2\theta = 0, \pi, 2\pi, \ldots$$
$$\theta = 0, \frac{\pi}{2}, \pi, \ldots$$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (3 \sin 2\theta)^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9 \sin^2 2\theta \, d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4\theta) d\theta =$$

$$\frac{9}{4}\int_0^{\frac{\pi}{2}} (1 - \cos 4\theta) d\theta = \frac{9}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{9}{4}\left(\frac{\pi}{2}\right) = \frac{9\pi}{8}$$

Multiply by 4 to get the area bounded by the entire graph.

Example.
Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.



$$2 + 2\sin\theta = 6\sin\theta$$

$$2 = 4\sin\theta$$
$$\frac{1}{2} = \sin\theta$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\sin\theta)^2 - (2 + 2\sin\theta)^2 d\theta = \frac{2}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (6\sin\theta)^2 - (2 + 2\sin\theta)^2 d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 36\sin^2\theta - (4 + 8\sin\theta + 4\sin^2\theta)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 36\sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta\, d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 32\sin^2\theta - 4 - 8\sin\theta\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 32\left(\frac{1}{2}\right)(1 - \cos 2\theta) - 4 - 8\sin\theta\, d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 16(1 - \cos 2\theta) - 4 - 8\sin\theta\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 16 - 16\cos 2\theta - 4 - 8\sin\theta\, d\theta =$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12 - 16\cos 2\theta - 8\sin\theta\, d\theta = 12\theta - 8\sin 2\theta + 8\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$12\left(\frac{\pi}{2}\right) - 12\left(\frac{\pi}{6}\right) - 0 + 0 + \frac{8\sqrt{3}}{2} - 8\frac{\sqrt{3}}{2} = 6\pi - 2\pi = 4\pi$$

Next time: conic sections.
Review conic sections in rectangular coordinates.