11/17/2022

Calculus on vectors and parametric equations (7.2)

The mechanics of calculus on a set of parametric equations or on a vector are quite straight forward. The issue will be how do we relate the derivative in parametric form to the traditional derivative or calculus concepts.

Example.

Find the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for the set of parametric equations $x(t) = t^2 + 1, y(t) = e^t - 6$

$$x'(t) = \frac{dx}{dt} = 2t$$

$$y'(t) = \frac{dy}{dt} = e^t$$

In vector form, we take the derivative component by component.

$$\vec{r}(t) = \langle t^2 + 1, e^t - 6 \rangle$$

$$\vec{r}'(t) = \langle 2t, e^t \rangle$$

What is the slope of the tangent line to a curve in parametric form?

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}}$$

Let's do that for our curve.

$$\frac{dy}{dx} = \frac{e^t}{2t}$$

What is slope of the tangent line at $t = 1$?

$$\frac{dy}{dx}(t = 1) = \frac{e}{2}$$

Equation of the tangent line at $t = 1$, to get the (x,y) that goes into the line, plug t=1 into the original set of equations.

$$x(1) = 2$$
$$y(1) = e - 6$$

$$(y - y_1) = m(x - x_1)$$

$$y - (e - 6) = \frac{e}{2}(x - 2)$$

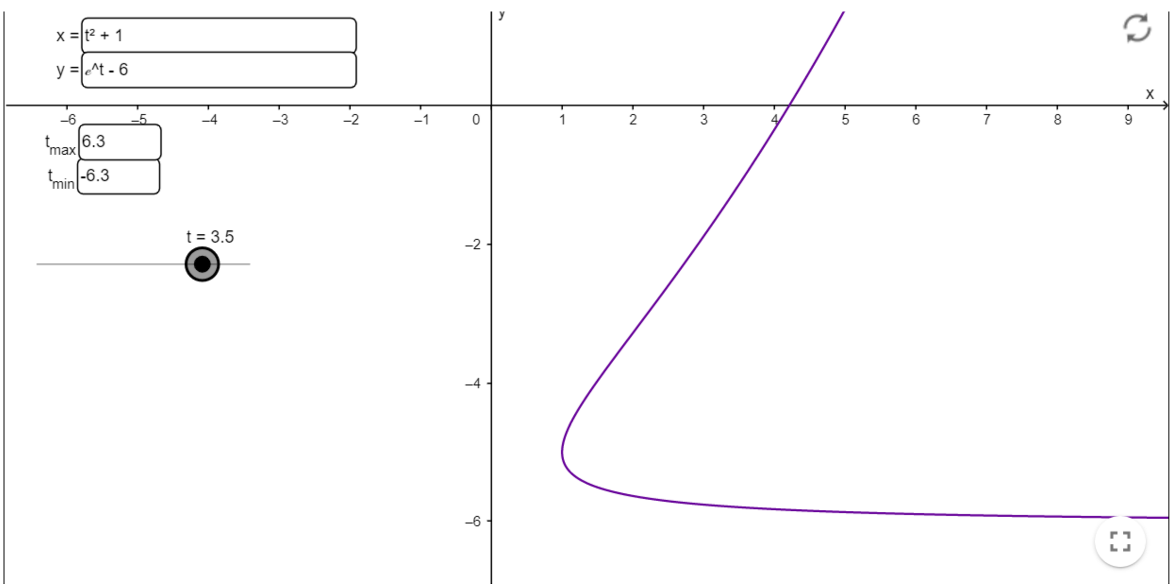
Can solve that for y.

What are the critical points of the curve (if any)? Where is the slope of the tangent line horizontal? (Recall, this is where the extrema are.) $\frac{dy}{dx} = 0$ and that implies that $\frac{dy}{dt} = 0$.

In this example $\frac{dy}{dx} = \frac{e^t}{2t}$ and we can conclude that there are no horizontal tangents, because $\frac{dy}{dt} = e^t$ which is never 0.

Where is the slope of the tangent line vertical?
Vertical lines have a slope which is undefined, and so that can happen where there is a zero in the denominator: $\frac{dx}{dt} = 0$.

In this example, the tangent line is vertical when $t = 0$.



What about the second derivative?

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right] = \frac{\frac{d}{dt}\left[\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right]}{\frac{dx}{dt}}$$

Procedurally: take the first derivative and find $\frac{dy}{dx}$ as above.

This gives us an expression for $\frac{dy}{dx}$ as a function of t.

Now, take the derivative of that $\frac{dy}{dx}$ with respect to t, and the divide the result by another copy of $\frac{dx}{dt}$

Example.

$$x(t) = t^2 + 1, y(t) = e^t - 6$$

$$x'(t) = 2t, y'(t) = e^t$$

$$\frac{dy}{dx} = \frac{e^t}{2t}$$

Take the derivative of the first derivative with respect to t.

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{e^t}{2t}\right] = \frac{e^t(2t) - 2e^t}{4t^2} = \frac{2e^t(t-1)}{4t^2} = \frac{e^t(t-1)}{2t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left[\frac{e^t(t-1)}{2t^2}\right]}{2t} = \frac{e^t(t-1)}{4t^3}$$

Recall the second derivative gives us concavity.

Where does the second derivative change sign? One sign change takes place at t=0, and there is a sign change when t=1. Does this agree with the graph? Yes, this is consistent with the graph. The concavity is upward (+) for t<0. It's negative (downward) for 0<t<1, and then positive (upward) again for t>1.

$t = 0, t = 1$ are inflection points.

If we want to take more derivatives, we can by continuing to take the derivative with respect to t of the previous derivative and then dividing by dx/dt again each time.

What about integration? How do we find the area under a curve defined parametrically?

Given: a parametric curve is defined by $x(t)$ and $y(t)$ on some interval, and the curve does not intersection with itself on that interval.

$$A = \int_a^b y(t)x'(t)dt = \int_a^b y(t)\frac{dx}{dt}dt$$

Find the area under the curve $x(t) = t^2 + 1, y(t) = e^t - 6$ on the interval $(0,1)$.

$$A = \int_0^1 (e^t - 6)\ 2t\ dt = \int_0^1 2te^t - 12t\ dt =$$

$$2te^t - \int_0^1 2e^t dt - 6t^2 = 2te^t - 2e^t - 6t^2|_0^1 = 2e - 2e - 6 - 0 + 2 + 0 = -4$$

$$u = 2t, dv = e^t dt$$
$$du = 2dt, v = e^t$$

The negative sign is consistent with the graph because the curve is under the x-axis, and so the signed area is negative.

Arc length?
In parametric form

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In vector form: recall that $\vec{r}(t) = \langle x(t), y(t) \rangle$ and the derivative was $\vec{r}'(t) = \langle x'(t), y'(t) \rangle$. What is the length of the derivative vector?

$$\|\vec{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$s = \int_a^b \|\vec{r}'(t)\|\, dt$$

These are really the same formula, but one is in parametric form, and one is in vector form.

Example. We can set up the arc length formula for our parametric curve example. Use (0,1) as the interval.

$$s = \int_0^1 \sqrt{(2t)^2 + (e^t)^2}dt = \int_0^1 \sqrt{4t^2 + e^{2t}}dt$$

This will need to be integrated numerically.

Sometimes you need to use symmetry to get the arc length.

For instance, suppose you want to find the length of the circumference of a circle.

$$x(t) = 3\cos(t)\,, y(t) = 3\sin(\mathrm{t})$$

The length of the circumference is the arc length one time around the circle.

$$s = \int_0^{2\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}dt = \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt =$$

$$\int_0^{2\pi} \sqrt{9(\sin^2(t) + \cos^2(t))}dt = \int_0^{2\pi} 3\sqrt{1}\, dt = 3t|_0^{2\pi} = 3(2\pi) = 6\pi = 2\pi r$$

$$s = 4\int_0^{\frac{\pi}{2}} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}dt$$

We’re just finding the arc length in the first quadrant, and then multiplying by 4 to go all the way around the circle.

If you get 0 for an arc length, this is a good sign that you need to invoke symmetry to find the length of the curve.

This especially happens with trig functions that simplify to square functions that cancel with the square root (and should leave behind an absolute value).

One of the other application related to arc length that we discussed earlier was surface area.

$$S = \int_a^b r(x)\sqrt{1 + [f'(x)]^2}dx$$

The $r(x)$ changed depending on the axis we were rotating around.

In parametric form:

$$S = \int_a^b R(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt$$

$R(t)$ changes to either $x(t)$ as you rotate around the y-axis, or $y(t)$ as you rotate around the x-axis.

If you wanted to find the surface area of a sphere, you can use this, but you will need to invoke symmetry using only half or a quarter of the circle.

Next time we’ll talk about polar coordinates.