11/10/2022

Differential Equations, continued
Euler's method, continued
Separable Equations/Logistic Equations

We left off with Euler's Method

The basis of Euler's method is basically calc 1: we approximate the tangent line at the point where we start, and we move along for a short distance. The new point in space has a new slope based on the differential equation, we re-approximate the slope of the new tangent line at that point, and repeat. Some assumptions for accuracy are that the curve (to the solution) is smooth and we don't have vertical asymptotes (vertical tangents) near where were are approximating.

Start at $(x_0, y_0) \leftrightarrow y(x_0) = y_0$, Then we take an iterative step of approximating the tangent from the slope.

Where $y' = f(x, y)$

$$m_0 = f(x_0, y_0)$$

$$y_1 = m_0(\Delta x) + y_0$$

Repeat:

$$m_n = f(x_n, y_n)$$

$$y_{n+1} = m_n(\Delta \mathrm{x}) + y_n$$
$$x_{n+1} = x_n + \Delta x$$

$$\Delta x = h = \frac{x_n - x_0}{n}$$

We have the differential equation $y' = y(3 - xy)$, and we want to approximate the solution to the differential equation after 10 steps if $y(0) = 1, and\ \Delta x = 0.1$.

We have the differential equation $y' = y(3 - xy)$, and we want to approximate the solution to the differential equation at $y(1)$ with 10 steps starting at $y(0) = 1$.

$$(x_0, y_0) = (0,1)$$

$$m_0 = 1\big(3 - 0(1)\big) = 3$$
$$y_1 = 3(0.1) + 1 = 1.3$$

New position:

$$(x_1, y_1) = (0.1, 1.3)$$

$$m_1 = (1.3)\big(3 - (0.1)(1.3)\big) = 3.731$$
$$y_2 = 3.731(0.1) + 1.3 = 1.6731$$

New position:

$$(x_2, y_2) = (0.2, 1.6731)$$

$$m_2 = (1.6731)\big(3 - 0.2(1.6731)\big) = 4.4594 \dots$$
$$y_3 = 4.4594 \dots (0.1) + 1.6731 = 2.11904 \dots$$

New position:

$$(x_3, y_3) = (0.3, 2.11904 \dots)$$

Continue in this process until we get to $(1, y_{10})$

Rest of the calculations in the Excel file.

Separable Differential Equations – these are first order equations that primarily depend on our integration techniques in order to solve. We can algebraically separate the y variables to one side of the equation from the x variables that are left on the other side.

$$y' = \frac{dy}{dx} = f(x, y) = g(x)\big(h(y)\big)$$

We can solve for $y(x)$ by $\frac{dy}{h(y)} = g(x)dx$

Then

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

Usually, this leaves us with an implicit solution and we may or may not be able to solve for y explicitly (as a function), but it gives us an expression for the shape of the curve for any initial condition.

Example.
Solve the differential equation

$$y' = \frac{\sqrt{1 - y^2}}{x - 1}$$

$$\frac{dy}{dx}(dx) = \frac{\sqrt{1 - y^2}}{x - 1}(dx)$$

$$\frac{1}{\sqrt{1 - y^2}} \times dy = \frac{\sqrt{1 - y^2}}{x - 1} dx \times \frac{1}{\sqrt{1 - y^2}}$$

$$\frac{dy}{\sqrt{1 - y^2}} = \frac{dx}{x - 1}$$

The variables are now separated

Integrate on each side with the corresponding variable.

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x-1}$$

$$\arcsin(y) = \ln(x-1) + C$$

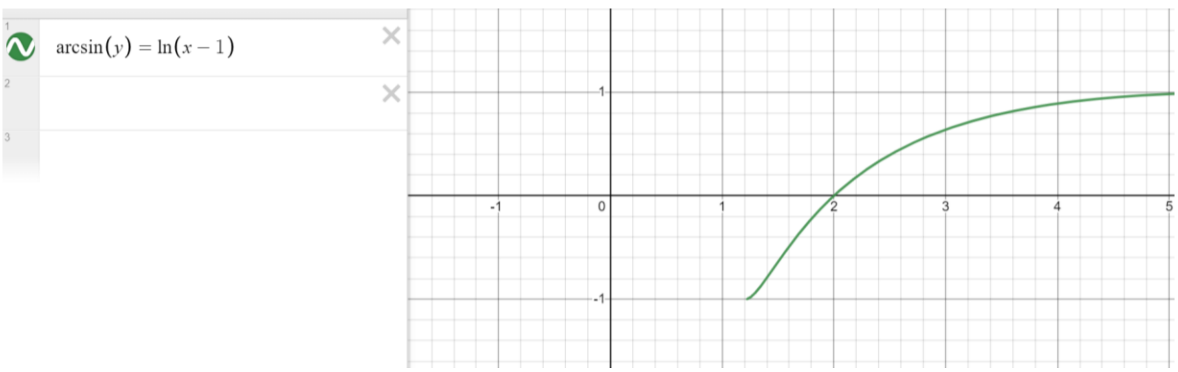
$$y = \sin(\ln(x-1) + C)$$

$$y(2) = 0$$

$$\arcsin(0) = \ln(2-1) + C$$

$$0 = 0 + C \rightarrow 0$$

$$\arcsin(y) = \ln(x-1)$$



Example.

A typical population growth problem, the population grows in proportion to the current population.

$$y' = ky$$

$$\frac{dy}{dx} = ky$$

$$\frac{dy}{y} = kdx$$

$$\int \frac{1}{y} dy = k \int dx$$

$$\ln|y| = kx + C$$

$$y = e^{kx+C} = e^{kx}e^{C} = y_0 e^{kx}$$

$$y(x) = y_0 e^{kx}$$

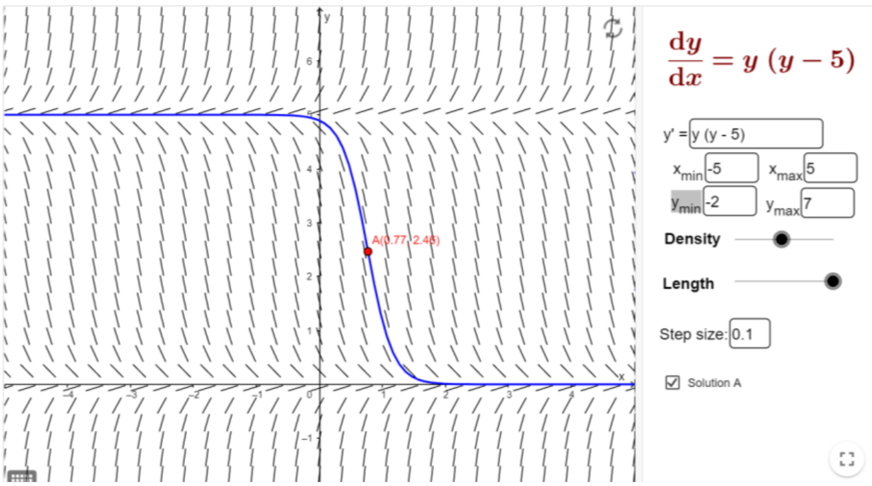
Logistic population models:

Derive from autonomous differential equations: (no explicit independent variable)

$$y' = f(y)$$

$$y' = y(y - 5)$$

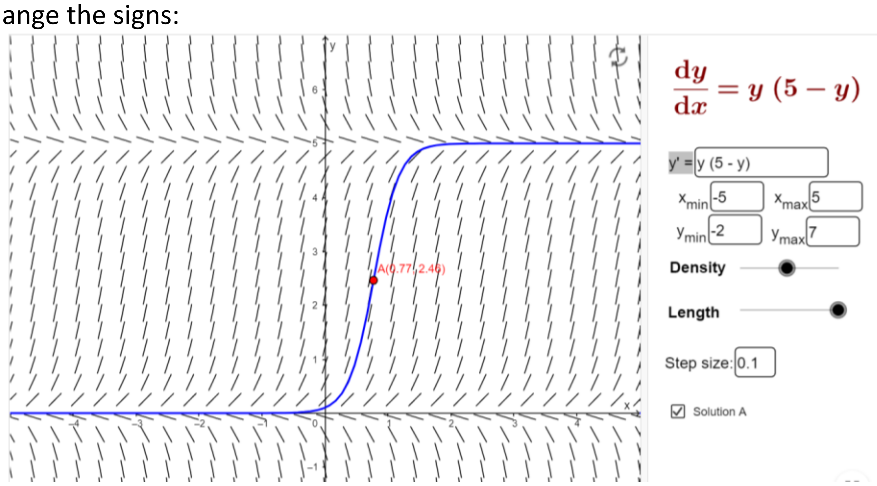
The direction fields don't depend on x, and so the slope is the same everywhere for the same y-value.



Equilibria where the slope is 0 for a constant y value. Here, y=0, y=5.
This s-curve shape between the equilibria, and exponential (approx.) above or below the equilibria.

Y=5 in this model is a threshold (extinction line). The population will grow above this line, but if you fall below the line, then the population will collapse to 0. (unstable equilibrium)

If we change the signs:



In this case, y=5 is called a carrying capacity (think of it as the maximum sustainable population for the area's resources). You grow up to this equilibrium, or if above, fall back to it. (stable equilibrium)

$$y' = y(y-5)$$

$$\frac{dy}{y(y-5)} = dx$$

In order to integrate the left side of the equation, I need to apply partial fractions.

$$\frac{A}{y} + \frac{B}{y-5} = \frac{1}{y(y-5)}$$

$$Ay - 5A + By = 1$$

$$A + B = 0$$
$$-5A = 1$$
$$A = -\frac{1}{5}$$
$$B = \frac{1}{5}$$

$$\int \frac{-\frac{1}{5}}{y} dy + \int \frac{\frac{1}{5}}{y-5} dy = \int dx$$

$$-\frac{1}{5}\ln(y) + \frac{1}{5}\ln(y-5) = x + C$$

$$\ln(y-5) - \ln(y) = 5x + C$$

$$\ln\left(\frac{y-5}{y}\right) = 5x + C$$
$$\frac{y-5}{y} = Ae^{5x}$$

$$y - 5 = yAe^{5x}$$

$$y - yAe^{5x} = 5$$

$$y(1 - Ae^{5x}) = 5$$

$$y(x) = \frac{5}{1 - Ae^{5x}}$$

The value in the numerator is related to the carrying capacity/threshold value
A is related to the initial condition relative to the carrying capacity.

This is the kind of function that produces our s-curve.

Next week: we start parametric equations/vectors.