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Series Tests, continued

Geometric Series Test

If a series is geometric $\sum_{n=0}^{\infty} a_0(r)^n$, then if $|r| < 1$ the series converges, and if $|r| \geq 1$, the series diverges.

$$a_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$r = \frac{1}{2} < 1$ so we can say that the series converges.

Moreover, have a formula for the sum, if the series converges:

$$S_\infty = \frac{a_0}{1 - r}$$

$$S = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Example.

$$a_n = \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$$

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} = \sum_{n=0}^{\infty} \frac{(-3)^n(-3)(4)}{4^{n-1}(4)} = \sum_{n=0}^{\infty} \frac{(-3)^n(-12)}{4^n} = \sum_{n=0}^{\infty} (-12)\left(-\frac{3}{4}\right)^n$$

$\left|-\frac{3}{4}\right| < 1$ so the series converges.

The infinite sum is $S = -\frac{12}{1 - -\frac{3}{4}} = -\frac{12}{\frac{7}{4}} = -12\left(\frac{4}{7}\right) = -\frac{48}{7}$

Telescoping Series

$$a_n = \sum_{n=1}^{\infty} \frac{a}{(n+b)(n+c)}$$

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Apply partial fractions to split these into two terms.

$$\frac{A}{n} + \frac{B}{n+1} = \frac{1}{n(n+1)}$$

$$A(n+1)+B(n)=1$$
$$An+A+Bn=1$$

$$A+B=0, A=1, B=-1$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$S_1=\frac{1}{1}-\frac{1}{2}=1-\frac{1}{2}$$

$$S_2=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3}$$

$$S_3=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=1-\frac{1}{4}$$

$$S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$$

What is the infinite sum?

$$S_\infty=a_1-\lim_{n\to\infty}a_{n+1}=a_1-\lim_{n\to\infty}\frac{1}{n+1}=1-0=1$$

If the factors in the denominator differ by 2

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+2}\right]=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots++\left(\frac{1}{n-2}-\frac{1}{n}\right)$$
$$+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac{1}{n}-\frac{1}{n+2}\right)=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$$

$$S=a_1+a_2+\lim_{n\to\infty}\left[-\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}=\sum_{n=1}^{\infty}\left[\frac{\frac{1}{2}}{2n+1}-\frac{\frac{1}{2}}{2n+3}\right]=\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{2n+1}-\frac{1}{2n+3}\right]$$

$$\frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots\right]$$

Test here is basically, after applying partial fractions, does the final term in the series have a limit that is finite? If yes, then converges. If no, diverges.

$$a_n = \sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=2}^{\infty} \ln n - \ln(n+1)$$

$$S_n = (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + (\ln 4 - \ln 5) + \cdots = \ln 2 - \lim_{n\to\infty} \ln(n+1)$$

This series diverges.

This limit goes to infinity.

In the textbook, they start their geometric series at n=1, and then their powers are n-1 in the formula.

Test for Divergence/Nth-term test
If the limit of the sequence of sums for our series do not have a limit of 0, then the sum of the series diverges.

If we have

$$\sum_{n=1}^{\infty} a_n$$

And $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

Conversely, if $\lim_{n\to\infty} a_n = 0$, it doesn't tell you anything, the test is inconclusive. The series might converge, or it might diverge.

Integral Test
Cannot determine the value of the sum, but if we calculate the area under the curve as an estimate (think Reimann sums with $\Delta x = 1$), then if the integral (area) converges, then the sum converges, and if the area (integral) diverges, then the sum diverges.

$\sum_{n=1}^{\infty} a_n$ and $a_n = f(n)$, then if $\int_1^{\infty} f(x)dx$ converges, then so does the sum. If it diverges, so does the sum.

Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ vs. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1}{n} = 0, \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\int_1^{\infty} \frac{1}{x} dx = \ln(x)|_1^{\infty} = \lim_{n\to\infty} \ln(n) - \ln(1) = \infty$$

$$\int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_1^{\infty} = -\lim_{n\to\infty} \frac{1}{n} - \left(-\frac{1}{1}\right) = 1$$

The harmonic series 1/n diverges, but the $\frac{1}{n^2}$ series converges.

P-series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Test: if p>1, the series converges. If $p \leq 1$, the series diverges.

Ex. $\int \frac{1}{x^{\frac{1}{2}}} dx = \int x^{-\frac{1}{2}} dx$ if you integrate, the power rule makes the exponent positive. $x^{1/2}$ when you plug in infinity, this positive power will go to infinity. This is true any time the exponent is less than 1.

Ex. $\int \frac{1}{x^{1.1}} dx = \int x^{-1.1} dx$ which will become $x^{-0.1}$ which is still a negative exponent, and so when you put in infinity, you are still dividing by a big number it converges.

Error on the integral test, when $a_n = f(n)$

$$E \leq \int_N^{\infty} f(x)\, dx$$

If you want to estimate the number of terms required to achieve a particular error, then use the resulting formula to solve for N (and then round to the next larger integer).