10/27/2022

Applications of Taylor and Power Series
Review for Exam #2

If I wanted the first couple of terms for the power series $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \approx 1 + x + x^2 + x^3 + x^4 + \cdots$
And if we integrate… $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots$

$$\int \sum_{k=0}^{\infty} x^k \, dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

Application of Taylor Series and Power Series

1) Take derivatives more easily.
2) Do integration of functions that we don't have simple antiderivatives for, or even any "analytic" solutions for.
3) Can make more complex functions from base function: include multiplication and division of power series
4) Simplify and find limits of complex expressions.
5) Not in this class—we can use power series to find solutions to differential equations that don't have any other solution techniques (this is an important for quantum mechanics).

Make more complex functions using substitution.

Once I have a Taylor series/power series formula for a base function (like a library of function), we can use substitution to derive more complex versions of the same type of function.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

What if we wanted to derive the formula for $f(x) = e^{-2x}$? Can we derive that from the e^x formula?

Yes.

Replace everywhere there is an x with (-2x) in the formula.

$$e^{-2x} = \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k x^k}{k!}$$

That is the same formula we derived in the last class.

What if we wanted to derive a Taylor series for $f(x) = \sin(x-3)$?

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Replace x with x-3

$$\sin(x-3) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-3)^{2k+1}}{(2k+1)!}$$

What if we wanted to create the power series for e^{x^2}?

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Replace x with x^2

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

We've said before there is no closed form solution (no analytic solution) for $\int e^{x^2} dx$, but we can find a Taylor series for the integral.

$$\int e^{x^2} dx = \int \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} dx = \sum_{k=0}^{\infty} \int \frac{x^{2k}}{k!} dx = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!\,(2k+1)} + C$$

Recall that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$$

$$e^x - e^{-x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!} - \frac{(-1)^k x^k}{k!} \right) =$$

$$(1-1) + (x - (-1)x) + \left(\frac{x^2}{2} - \frac{x^2}{2} \right) + \left(\frac{x^3}{6} - \frac{(-1)x^3}{6} \right) + \left(\frac{x^4}{24} - \frac{x^4}{24} \right) + \left(\frac{x^5}{120} - \frac{(-1)x^5}{120} \right) + \cdots$$

$$0 + 2x + 0 + \frac{2x^3}{6} + 0 + \frac{2x^5}{120} + \cdots$$

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots \right)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Multiplying Taylor/power series

$$e^x \sin(x) = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) \approx$$

$$(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots)(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)$$

Suppose we to find P_4 approximation of $f(x) = e^x \sin(x)$. Write the terms of this series.
FOIL term by term, stop/drop terms when the product exceeds the required power…

$$1\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = x - \frac{x^3}{6} + \cdots$$

$$x\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = x^2 - \frac{x^4}{6} + \cdots$$

$$\frac{x^2}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = \frac{x^3}{2} - \cdots$$

$$\frac{x^3}{6}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = \frac{x^4}{6} - \cdots$$

$$\frac{x^4}{24}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) = \cdots$$

$$e^x \sin(x) \approx P_4 = x + x^2 + \left(-\frac{x^3}{6} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{6} + \frac{x^4}{6}\right) = x + x^2 + \frac{x^3}{3} + \cdots$$

Division with Taylor/Power series.

Suppose I want a Taylor series for $f(x) = \frac{\sin(x)}{1-x}$? Find P_4 for this function.

$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$
\begin{array}{r}
x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4 \cdots \\
1 - x \,\big)\overline{\, 0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5 + 0x^6} \\
-x + x^2 \\
\hline
x^2 - \frac{1}{6}x^3 - \cdots \\
-x^2 + x^3 \\
\hline
\frac{5}{6}x^3 + 0x^4 \\
-\frac{5}{6}x^3 + \frac{5}{6}x^4 \\
\hline
\frac{5}{6}x^4 + \frac{1}{120}x^5 \\
-\frac{5}{6}x^4 + \frac{5}{6}x^5 \\
\hline
\end{array}
$$

$$(x) = \frac{\sin(x)}{1-x} \approx x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4 + \cdots$$

Likewise, we can use Taylor Series to simplify complex limit expressions instead of doing L'Hopital's.

$$\frac{\sin(x)}{x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$$

Exam #2
Covers two chapters: Chapters 5 and 6 in online text
Sequences (do not forget about these!!!) and Series Tests.
Power Series and Taylor Series

Same format as last time. Online submission for Part 1, and hand-graded submission (like the quizzes) for Part 2.

Part 1: Sequence limits, convergence/divergence of series, sum of geometric/telescoping series, error calculations (integral test/alternating series test), intervals of convergence (radius of convergence), errors on Taylor series.

Part 2: work for series convergence, power series functions, Taylor series derivations, applications of power/Taylor series, intervals of convergence.

Table of Common Taylor/Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$