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Taylor Polynomials/Taylor Series

Polynomial vs. Series
Polynomial has a highest degree term (stop counting at n), whereas, a Taylor series goes on to infinity. A Taylor polynomial is essentially an approximation of Taylor series, where just stop counting after a specified number of terms.

P_4: The 4th degree approximation of the infinite series, spelled out with a list of terms
P_∞: the equivalent notation for the infinite series: summation notation generally.

Taylor series is the general name for power series derived from the process of taking derivatives; they can be centered anywhere.

Maclaurin Series: these are Taylor series, but centered at 0

Taylor series is a power series, but the original function is not required to be rational function (as was the case with the geometric series method of creating power series). Instead, we going to extend the process of linear approximations to higher order derivatives in order to obtain a polynomial approximation for a function.

Polynomials are easy to work with. It's easier to do proofs on polynomials than on other kinds of functions. The algebra is easier with polynomials, and by taking more and more terms of our approximation, we can make it as accurate as we like.

We can use the same procedure to recenter a polynomial (refactoring). Suppose you have an expression like $f(x) = x^3 + 2x^2 - 6x + 5$, but you want to rewrite the expression in terms of $(x-2)$ instead of just x.

Given $f(x)$ is a function, and it has continuous derivatives on some open interval near c. The Taylor series (approximation) is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

c is the center
$f^{(n)}$ is the nth derivative
$f^{(n)}(c)$ is the nth derivative evaluated at c
$n!$: is 4!=4(3)(2)(1)

$f^{(0)}(x) = f(x)$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{6} + \frac{f^{(4)}(c)(x-c)^4}{24} + \cdots$$

$$f^{(4)}(x) = f^{IV}(x)$$

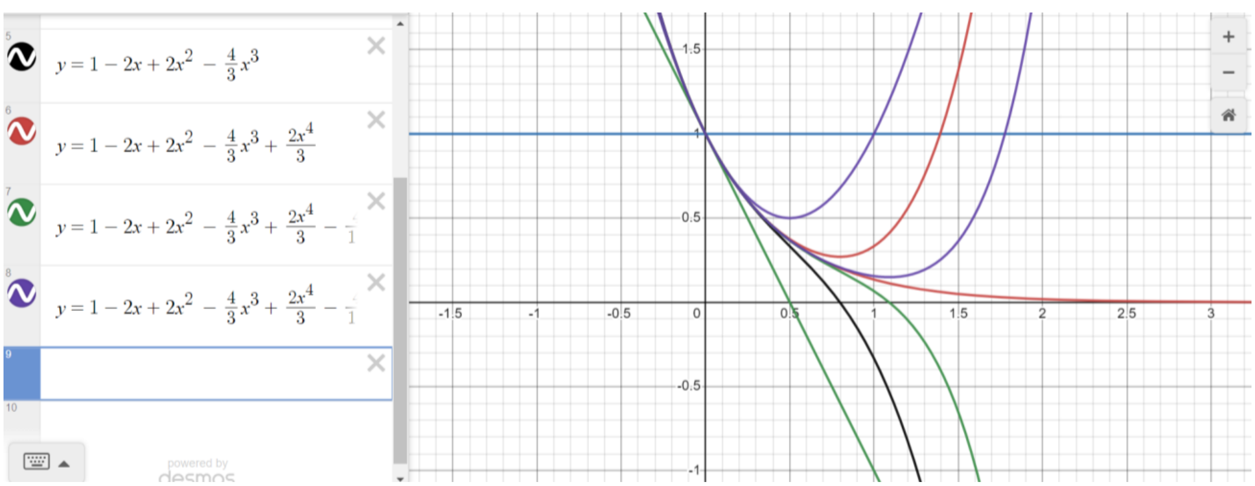
Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{6} + \frac{f^{(4)}(0)x^4}{24} + \cdots$$

Example. Find the Taylor series for the function $f(x) = e^{-2x}$ (centered at x=0; i.e. this will be a Maclaurin series). Plot $P_0, P_1, P_2, \ldots, P_6$ with the original function.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)(x-c)^n}{n!}$
0	1	e^{-2x}	1	1	$\frac{(1)(1)}{1} = 1$
1	1	$-2e^{-2x}$	-2	x	$-\frac{2x}{1} = -2x$
2	2	$4e^{-2x}$	4	x^2	$\frac{4x^2}{2} = 2x^2$
3	6	$-8e^{-2x}$	-8	x^3	$-\frac{8x^3}{6} = -\frac{4}{3}x^3$
4	24	$16e^{-2x}$	16	x^4	$\frac{16x^4}{24} = \frac{2x^4}{3}$
5	120	$-32e^{-2x}$	-32	x^5	$-\frac{32x^5}{120} = -\frac{4}{15}x^5$
6	720	$64e^{-2x}$	64	x^6	$\frac{64x^6}{720} = \frac{4}{45}x^6$
7	5040	$-128e^{-2x}$	-128	x^7	$-\frac{128x^7}{5040} = -\frac{8}{315}x^7$

$$f(x) = e^{-2x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$$



$$P_0 = 1$$
$$P_1 = 1 - 2x \ (linear\ approximation)$$

$$P_2 = 1 - 2x + 2x^2$$
$$P_3 = 1 - 2x + 2x^2 - \frac{4}{3}x^3$$
$$P_4 = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2x^4}{3}$$
$$P_5 = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2x^4}{3} - \frac{4}{15}x^5$$
$$P_6 = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2x^4}{3} - \frac{4}{15}x^5 + \frac{4}{45}x^6$$

What we see is that as we extend the Taylor polynomial to more terms, we get better and better approximations to the curve, and they stay closer to the original function for longer.

Most books will include a table of common Taylor series. $e^x, \sin(x), \cos(x), \ln(x), \arctan(x), \frac{1}{x}, \ldots$ Sometimes it includes $\tan(x)$, rational powers/binomial, and others like $\cosh(x)$.

Error on the Taylor series.

$$R_n \le \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right|$$

When we are estimating, the $f^{(n+1)}(z)$ we can think of as the maximum of the function on some interval where we are doing the approximation.

While this z value can in theory be anywhere in the interval where were are approximating, typically, we use the endpoints of the interval as long as the function is increasing or decreasing over the entire interval.

Consider our P_6 polynomial from our example. Since we stopped counting at n=6, the error/remainder is based on the 7th term.

$$R_n \approx \left| \frac{f^{(7)}(z)}{7!} x^7 \right|$$

$$f^{(7)}(x) = -128e^{-2x}$$
$$\left|f^{(7)}(x)\right| = 128e^{-2x}$$

We want to find the maximum error on the interval [-1,1].
This is a decreasing function on this interval with no critical points.
$f^{(7)}(-1) = 128e^{-2(-1)} = 128e^2$ (maximum)
$f^{(7)}(1) = 128e^{-2(1)} = 128e^{-2}$

Use the maximum in error estimate:

$$R_n \le \left| \frac{128e^2(-1)^7}{7!} \right| \approx 0.1876 \ldots$$

We want to find the maximum error on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

$f^{(7)}\left(-\frac{1}{2}\right) = 128e^{-2\left(-\frac{1}{2}\right)} = 128e$ (maximum)
$f^{(7)}\left(\frac{1}{2}\right) = 128e^{-2\left(\frac{1}{2}\right)} = 128e^{-1}$

Use the maximum in error estimate:

$$R_n \leq \left|\frac{128e\left(-\frac{1}{2}\right)^7}{7!}\right| \approx 5.39 \times 10^{-4}$$

Now that we have these Taylor approximations, what can we do with it?
Try to derive power series for more complex functions (at least polynomial approximations). Integrating and differentiating. Limits. Substitutions to create series for more complex functions.

$$\sum_{n=0}^{\infty} \frac{(2n)!\,x^n}{n^{2n}}$$

$$\lim_{n\to\infty} \frac{(2n+2)!\,x^{n+1}}{(n+1)^{2n+2}} \times \frac{n^{2n}}{(2n)!\,x^n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!\,x}{(n+1)^{2n+2}} \times \frac{n^{2n}}{(2n)!} =$$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)x}{(n+1)^{2n+2}} \times \frac{n^{2n}}{1} = \lim_{n\to\infty} \frac{2(n+1)(2n+1)x}{(n+1)^{2n}(n+1)^2} \times \frac{n^{2n}}{1} =$$

$$\lim_{n\to\infty} \frac{2(2n+1)x}{(n+1)^{2n}(n+1)} \times \frac{n^{2n}}{1} = \left(\lim_{n\to\infty} \frac{2(2n+1)}{(n+1)}\right)\left(\lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}}\right)\left(\lim_{n\to\infty} |x|\right)$$

$$\lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}}$$

Do the reciprocal to make the math easier:

$$\lim_{n\to\infty} \frac{(n+1)^{2n}}{(n)^{2n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n} = e^2$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$ln\left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right) = \ln{(L)}$$

$$\lim_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n^{-1}} = \lim_{n\to\infty} \frac{\frac{1}{\left(1+\frac{1}{n}\right)}(-n^{-2})}{-n^{-2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$= \ln(L)$$

$$L = e$$

$$\left(\lim_{n\to\infty} \frac{2(2n+1)}{(n+1)}\right)\left(\lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n}}\right)\left(\lim_{n\to\infty} |x|\right) = 4(e^{-2})|x| < 1$$