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Power Series (6.1/6.2)

Creating power series for rational functions – or functions whose derivatives are rational functions (log functions and arctangent functions)

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} a(x^n) = \frac{a}{1-x}$$

$$\sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \frac{a(x)}{1-u(x)}$$

Starting with the rational function, and converting the rational function into a power series.

$$f(x) = \frac{a}{1-x}$$

Rewrite $f(x)$ as $\sum_{n=0}^{\infty} a(x^n)$, which will be our power series expression for the same function.

Suppose

$$f(x) = \frac{3}{1-x}$$

Write this function as a power series.

$$f(x) = \sum_{n=0}^{\infty} 3(x^n)$$

What we are claiming here is that for every value of x for which the power series converges, the sum of the power series for a particular value of x in the convergent interval will be the same as the value of the rational expression.

What if our rational function is a little more complicated?

$$f(x) = \frac{6x}{1+x^2}$$

How would I create a power series for this function?

$$\sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \frac{a(x)}{1-u(x)}$$

$$f(x) = \frac{6x}{1+x^2} = \frac{6x}{1-(-x^2)}$$

$$a(x) = 6x, u(x) = -x^2$$

$$f(x) = \sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \sum_{n=0}^{\infty} 6x(-x^2)^n = \sum_{n=0}^{\infty} 6x(-1)^n(x^{2n}) = \sum_{n=0}^{\infty} (-1)^n 6x^{2n+1}$$

$$\approx 6x - 6x^3 + 6x^5 - 6x^7 + 6x^9 - 6x^{11} + \cdots$$

Example.

$$f(x) = \frac{x}{3-5x} = \frac{\left(\frac{1}{3}\right)x}{\left(\frac{1}{3}\right)(3-5x)} = \frac{\frac{1}{3}x}{1-\frac{5}{3}x}$$

$$\sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \frac{a(x)}{1-u(x)}$$

$$a(x) = \frac{1}{3}x, u(x) = \frac{5}{3}x$$

$$f(x) = \sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \sum_{n=0}^{\infty} \frac{1}{3}x\left(\frac{5}{3}x\right)^n = \sum_{n=0}^{\infty} \frac{1}{3}x\frac{5^n}{3^n}x^n = \sum_{n=0}^{\infty} \frac{5^n}{3^{n+1}}x^{n+1}$$

For our simplest cases $f(x) = \frac{a}{1-x}$, match the rational function to the formula

1) Check the sign of the u(x) in the denominator, and
2) make the constant in the denominator a +1 (by multiplying as needed).

The u function cannot be "complicated" by having multiple terms in it. The most complicated u(x) you are allowed is of the form $(x-c)^p$

If you have an expression like $x^2 + 2x$ that can't be u(x).

Example.

$$f(x) = \frac{2x}{1+4x+x^2}$$

How do I turn this into a power series?

Complete the square

$$f(x) = \frac{2x}{(1-4)+(4+4x+x^2)} = \frac{2x}{-3+(x+2)^2} = \frac{\left(-\frac{1}{3}\right)2x}{\left(-\frac{1}{3}\right)(-3+(x+2)^2)} = \frac{-\frac{2}{3}x}{1-\frac{1}{3}(x+2)^2}$$

$$a(x) = -\frac{2}{3}x, u(x) = \frac{1}{3}(x+2)^2$$

$$f(x) = \sum_{n=0}^{\infty} a(x)\big(u(x)\big)^n = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)x\left(\frac{1}{3}(x+2)^2\right)^n = \sum_{n=0}^{\infty} -\frac{\frac{2}{3}x(x+2)^{2n}}{3^n} = \sum_{n=0}^{\infty} -\frac{2x(x+2)^{2n}}{3^{n+1}}$$

$$u = x+2$$
$$x = u-2$$

$$\sum_{n=0}^{\infty} -\frac{2(u-2)(u)^{2n}}{3^{n+1}} = \sum_{n=0}^{\infty}\left[-\frac{2u^{2n+1}}{3^{n+1}} + \frac{4u^{2n}}{3^{n+1}}\right] = \sum_{n=0}^{\infty} -\frac{2u^{2n+1}}{3^{n+1}} + \sum_{n=0}^{\infty} \frac{4u^{2n}}{3^{n+1}} =$$

$$\sum_{n=0}^{\infty} -\frac{2(x+2)^{2n+1}}{3^{n+1}} + \sum_{n=0}^{\infty} \frac{4(x+2)^{2n}}{3^{n+1}}$$

All of the expressions we've created so far, all assume the interval of convergence is centered at 0 (or some other natural point based on the expression itself.

In a power series, (x-c), the center of your interval of convergence will be at c.

In our first three examples, the center of the interval of convergence is at 0 since it's just x raised to some power. In the last example, the center of the interval of convergence naturally ended up at x= -2.

We can force the center of the interval of convergence to be at a particular location.

$$f(x) = \frac{3}{1-x}$$

Center the interval of convergence at x=2.

$$f(x) = \frac{3}{1-(x-2)-2} = \frac{3}{1-x+2-2} = \frac{3}{-1-(x-2)} = \frac{(-1)(3)}{(-1)\big(-1-(x-2)\big)} = \frac{-3}{1+(x-2)} =$$

$$\frac{-3}{1-\big(-(x-2)\big)}$$

Write as power series centered at x=2.

$$f(x) = \sum_{n=0}^{\infty} -3\big(-(x-2)\big)^n = \sum_{n=0}^{\infty} -3(-1)^n(x-2)^n = \sum_{n=0}^{\infty} (-1)^{n+1}3(x-2)^n$$

Recall that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ and $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

Will need to shift the center of the interval of convergence off zero in order to write a power series for it because the function is not defined at zero.

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{1}{1 + (x-1)} = \frac{1}{1 - \big(-(x-1)\big)}$$

Power series for $\frac{1}{x}$ centered at x=1, is

$$f(x) = \sum_{n=0}^{\infty} \big(-(x-1)\big)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

In order to get the power series for the $\ln(x)$ function, I take the antiderivative of the expression.

$$\int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} + C$$

The value of C is the value of x at the center of the interval of convergence: x=1… therefore C=0.

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Do the same thing for arctan(x).

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x)$$

$C = 0$ because arctan(0) is 0 (and 0 is the center of the interval of convergence).

So what if we have a rational expression like

$$f(x) = \frac{x}{(1+2x^2)^3}$$

Now what?

$$a(1-x)^{-1} = \frac{a}{1-x} = \sum_{n=0}^{\infty} a(x^n)$$

First derivative:

$$a(-1)(1-x)^{-2}(-1) = a(1-x)^{-2} = \frac{a}{(1-x)^2} = \sum_{n=1}^{\infty} anx^{n-1} = \sum_{k=0}^{\infty} a(k+1)x^k$$

Reindex by replacing $n = k+1$, so $n-1 = k+1-1 = k$

A formula for a power series for a rational expression with a squared denominator:

$$\frac{a}{(1-x)^2} = \sum_{k=0}^{\infty} a(k+1)x^k$$

More complex:

$$\frac{a(x)}{(1-u(x))^2} = \sum_{k=0}^{\infty} a(x)(k+1)(u(x))^k$$

Our example had a cube in the denominator (around the whole thing)

Continue taking derivatives:
2nd derivative:

$$a(1-x)^{-2} = \frac{a}{(1-x)^2} = \sum_{n=1}^{\infty} anx^{n-1}$$

$$a(-2)(1-x)^{-3}(-1) = \frac{2a}{(1-x)^3} = \sum_{n=2}^{\infty} an(n-1)x^{n-2} = \sum_{k=0}^{\infty} a(k+2)(k+1)x^k$$

Reindex by replacing $n = k+2$

$$\frac{2a}{(1-x)^3} = \sum_{k=0}^{\infty} a(k+2)(k+1)x^k$$

Or the more complex case:

$$\frac{2a(x)}{(1-u(x))^3} = \sum_{k=0}^{\infty} a(x)(k+2)(k+1)(u(x))^k$$

$$\frac{a(x)}{(1-u(x))^3} = \frac{1}{2}\sum_{k=0}^{\infty} a(x)(k+2)(k+1)(u(x))^k$$

$$f(x) = \frac{x}{(1+2x^2)^3}$$

Now we can set up our power series:

Based on the last formula:

$$a(x) = x, u(x) = -2x^2$$

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty} x(k+2)(k+1)(-2x^2)^k = \sum_{k=0}^{\infty} \frac{1}{2} x(k+2)(k+1)(-1)^k(2)^k x^{2k} =$$

$$\sum_{k=0}^{\infty} (-1)^k 2^{k-1}(k+2)(k+1)x^{2k+1}$$

Recall from last time, we can find the interval of convergence by doing a ratio test to determine for which values of x the series converges.

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^k(k+3)(k+2)x^{2k+3}}{(-1)^k 2^{k-1}(k+2)(k+1)x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{2(k+3)x^2}{k+1}\right| = 2x^2 < 1$$

$$x^2 < \frac{1}{2}$$

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Radius is $\frac{1}{\sqrt{2}}$. To finish finding the interval, check the endpoints!!!!

$$\sum_{k=0}^{\infty} (-1)^k 2^{k-1}(k+2)(k+1)\left(\frac{1}{\sqrt{2}}\right)^{2k+1} = \sum_{k=0}^{\infty} (-1)^k 2^{k-1}(k+2)(k+1)\left(\frac{1}{2}\right)^k\left(\frac{1}{\sqrt{2}}\right)^1$$

Diverges as k goes to infinity since they are in the numerator.
Likewise for the left endpoint.
So the interval of convergence $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Next time we do Taylor Series.