MTH 264, Exam #1, Fall 2022 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the area bounded by the graphs $x = 4 - y^2$ and $x = y - 2$. Sketch the region. (10 points)

$\int_{-3}^{2} (4-y^2) - (y-2)\, dy =$

$4 - y^2 = y - 2$

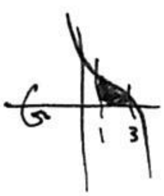
$0 = y^2 + y - 6$

$(y-2)(y+3)$

$\int_{-3}^{2} -y^2 - y + 6\, dy = -\frac{1}{3}y^3 - \frac{1}{2}y^2 + 6y \Big|_{-3}^{2} =$

$\frac{125}{6}$

2. Find the volume of the solid of revolution bounded by the graphs of $y = -(x-2)^3 + 2$, $y = 0$, $x = 1$ and $x = 3$, revolved around the x-axis using the disk or the washer method. (10 points)



$\pi \int_1^3 \left[-(x-2)^3 + 2\right]^2 dx = \pi \int_1^3 x^6 - 12x^5 + 60x^4 - 164x^3 + 264x^2 - 240x + 100\, dx$

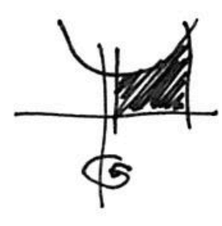
$= \pi\left[\frac{1}{7}x^7 - 2x^6 + 12x^5 - 41x^3 + \frac{264}{3}x^3 - 120x^2 + 100x\right|_1^3$

$= \frac{58\pi}{7}$

3. Find the volume of the solid of revolution bounded by the graphs of $y = (x-1)^2 + 3$, $x = 1$, $x = 5$, $y = 0$, and revolved around the y-axis, using the shell method. (10 points)

$2\pi \int_1^5 x\left[(x-1)^2 + 3\right] dx = 2\pi \int_1^5 x^3 - 2x^2 + 4x\, dx$

$= 2\pi\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + 2x^2\right|_1^5 = 2\pi\left[\frac{364}{3}\right]$



$= \frac{728\pi}{3}$

4. Find the arc length of the graph $y = 2x^{\frac{3}{2}} + 3$ on the interval [0, 9]. (8 points)

$$y' = \frac{3}{2}\cdot 2x^{1/2} = 3x^{1/2}$$

$u = 1+9x$
$du = 9dx$

$$\int_0^9 \sqrt{1+(3x^{1/2})^2}\,dx = \int_0^9 \sqrt{1+9x}\,dx$$

$$\int \sqrt{1+9x}\,dx = \int u^{1/2}\cdot\frac{1}{9}\,dx = \frac{1}{9}u^{3/2}\cdot\frac{2}{3} \to \frac{2}{27}[1+9x]^{3/2}\Big|_0^9 = \frac{164\sqrt{82}}{27} - \frac{2}{27}$$

5. On the interval [0,50], find the average value of the function $f(x) = -x^2 + 50x + 160$. (8 points) ≈ 54.929

$$\frac{1}{50}\int_0^{50} -x^2+50x+160\,dx = -\frac{1}{3}x^3+25x^2+160x\Big|_0^{50} = \frac{86500}{3}\cdot\frac{1}{50} = \frac{1730}{3}$$

6. Find the surface area of the surface of revolution generated by revolving the curve $y = 2\sqrt{x}$ over the interval [4,8] around the *x*-axis. (10 points)

$$y' = 2\cdot\frac{1}{2}x^{-1/2} = x^{-1/2} = \frac{1}{\sqrt{x}}$$

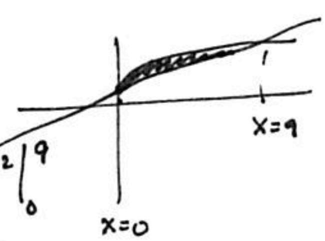
$$2\pi\int_4^8 2\sqrt{x}\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx = 2\pi\int_4^8 2\sqrt{x}\sqrt{1+\frac{1}{x}}\,dx =$$

$$2\pi\int_4^8 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 2\pi\int_4^8 \frac{2\sqrt{x}}{\sqrt{x}}\sqrt{1+x}\,dx = 2\pi\int_4^8 2\sqrt{1+x}\,dx$$

$$2\pi\left[2\cdot\frac{2}{3}(1+x)^{3/2}\Big|_4^8 = 2\pi\left[\frac{-4(5\sqrt{5}-27)}{3}\right] = \frac{-8(5\sqrt{5}-27)\pi}{3}\right.$$

$\approx 42.1858\pi \approx 132.53$

7. Find the centroid of a laminar sheet on constant density bounded by the graphs $y = \sqrt{x}+1, y = \frac{1}{3}x+1$. (16 points)



$\sqrt{x} + \cancel{1} = \frac{1}{3}x + \cancel{1}$ $\quad M = \rho\int_0^9 (\sqrt{x}+1) - (\frac{1}{3}x+1)\,dx =$

$\sqrt{x} = \frac{1}{3}x$ $\quad = \int_0^9 \sqrt{x} - \frac{1}{3}x\,dx = \frac{2}{3}x^{3/2} - \frac{1}{6}x^2\Big|_0^9 = 9/2$

$3\sqrt{x} = x$

$9x = x^2$ $\quad M_x = \frac{\rho}{2}\int_0^9 (\sqrt{x}+1)^2 - (\frac{1}{3}x+1)^2\,dx =$

$x^2 - 9x = 0$

$x(x-9) = 0$

$x=0, x=9$

$$\frac{1}{2}\int_0^9 x+2\sqrt{x}+1 - (\frac{1}{9}x^2+\frac{2}{3}x+1)\,dx = \frac{1}{2}\int_0^9 -\frac{1}{9}x^2+\frac{1}{3}x+2\sqrt{x}\,dx =$$

$$\frac{1}{2}\left[-\frac{1}{27}x^3+\frac{1}{6}x^2+\frac{4}{3}x^{3/2}\Big|_0^9 = \frac{1}{2}[45/2] = 45/4\right.$$

$$M_y = \rho\int_0^9 x(\sqrt{x}+1-\frac{1}{3}x-1)\,dx = \int_0^9 x^{3/2} - \frac{1}{3}x^2\,dx = \frac{2}{5}x^{5/2} - \frac{1}{9}x^3\Big|_0^9$$

$$= \frac{81}{5}$$

$\bar{x} = \frac{\cancel{81}^{9}}{5}\cdot\frac{2}{\cancel{9}} = \frac{18}{5}$ $\quad \bar{y} = \frac{45/4}{9/2} = \frac{\cancel{45}^{5}}{\cancel{4}_{2}}\cdot\frac{2}{\cancel{9}} = 5/2$ $\quad (18/5, 5/2)$

8. A force of 600 newtons stretches a spring 50 centimeters on a mechanical device for driving fence posts. Find the work done in compressing the spring an additional 20 centimeters. (10 points)

$600 = k(.5) \rightarrow k = 1200$

$$\int_{.5}^{.7} 1200x\,dx = 600x^2 \Big|_{0.5}^{0.7} = 144 \text{ N-m}$$

9. For the following integrals, state which method you would use, and which basic integration rule. Do not actually perform the integration. Methods may include: substitution, change of variables, complete the square, add/subtract, trig identities, long division, partial fractions, by parts, trig substitution, etc. Basic integration rules may include: power rule, log rule, exponential rule, trig functions, inverse trig functions, etc. Some problems may require more than more method or rule. (5 points each)

a. $\int \frac{3}{2-x^2}dx$ factor denom, partial fractions, log rule

b. $\int x\sqrt{x+1}dx$ change of variables or integration by parts, power rule

c. $\int x^3 e^{x^2} dx$ integration by parts + substitution power rule + exponential

d. $\int \frac{1}{\sqrt{7-x^2}}dx$ trig substitution, inverse trig functions

e. $\int \frac{1}{x^2+18x+82}dx$ complete the square, u-sub, arctangent rule

f. $\int \cos^2 x dx$ trig identity, trig integration

10. Use Trapezoidal Rule to approximate the area under the curve of $\int_1^2 e^x \ln x dx$ for n=6. (10 points)

$1, 7/6, 8/6, 9/6, 10/6, 11/6, 2$

$\frac{2-1}{6} = \frac{1}{6} = h$

$$\approx \frac{1}{6}\cdot\frac{1}{2}\left[e^1 \ln 1 + 2e^{7/6}\ln(7/6) + 2e^{8/6}\ln(8/6) + 2e^{9/6}\ln(9/6) + 2e^{10/6}\ln(10/6) + 2e^{11/6}\ln(11/6) + e^2\ln(2)\right]$$

≈ 2.07669

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

11. Use the definition of the hyperbolic sine function to prove that its derivative is the hyperbolic cosine function, i.e. that $\frac{d}{dx}[\sinh x] = \cosh x$. (15 points)

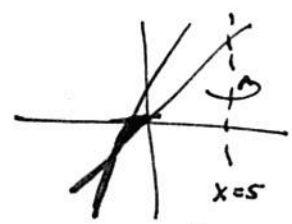
$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}[e^x - e^{-x}] = \frac{1}{2}[e^x + e^{-x}] = \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

12. Find the volume of the solid form from revolving the region formed by the equations $y = x, y = 2x+1, y = 0$ around the line x=5. Use the method of your choice. Sketch the region. (15 points)

$y - 1 = 2x \to x = \frac{y-1}{2}$

$x = 2x + 1$
$x = -1$



washer

$$\pi\int_{-1}^{0} \left[5 - \left(\frac{y-1}{2}\right)\right]^2 - [5 - y]^2\, dy =$$

$$\pi\int_{-1}^{0} -\frac{3y^2}{4} + \frac{9 \cdot y}{2} + \frac{21}{4}\, dy = \pi\left[-\frac{y^3}{4} + \frac{9y^2}{4} + \frac{21}{4}y\right]_{-1}^{0}$$

$$= \pi\left[\frac{11}{4}\right] = \frac{11\pi}{4}$$

13. Consider a cylindrical tank with diameter 14 feet and a height of 12 feet. Find the amount of work needed to drain the tank if it is only half full. Assume that the fluid in the tank is water and it has a weight-density of 62.4 lbs./ft³. (20 points)

$\cancel{A} = \cancel{\pi} \cdot 49$

$$\int_0^6 62.4 \cdot 49\pi\,(12 - y)\, dy =$$

$$62.4 \cdot 49\pi \int_0^6 12 - y\, dy = 62.4 \cdot 49\pi\left[12y - \frac{1}{2}y^2\right]_0^6$$

$$= 62.4 \cdot 49\pi\,[54] =$$

$$165110\pi$$

14. Set up (but do not solve) this rational expression $\frac{x^4-7x^2+8x-11}{(x^2+1)(x-2)^2(x^2+4)^2(x+3)}$ for decomposition by partial fractions. (16 points)

$$\frac{Ax+B}{x^2+1}+\frac{C}{x-2}+\frac{D}{(x-2)^2}+\frac{Ex+F}{x^2+4}+\frac{Gx+H}{(x^2+4)^2}+\frac{I}{x+3}$$

15. Integrate by an appropriate method. (10 points each)

a. $\int t\sqrt[3]{t-4}\,dt$ (note: this is a cube root, not a t^3)

$u=\sqrt[3]{t-4} \to u^3=t-4 \to u^3+4=t$

$3u^2\,du=dt$

$\int (u^3+4)u\cdot 3u^2\,du =$

$\int 3u^6+12u^3\,du = \frac{3}{7}u^7+3u^4+C = \frac{3}{7}(t-4)^{7/3}+3(t-4)^{4/3}+C$

b. $\int e^x \sin x\,dx$

$u=\sin x \quad dv=e^x dx$

$du=\cos x\,dx \quad v=e^x$

$e^x\sin x-\int e^x\cos x\,dx$

$u=\cos x \quad dv=e^x dx$

$du=-\sin x\,dx \quad v=e^x$

$e^x\sin x - e^x\cos x - \int \sin x\cdot e^x\,dx = \int e^x \sin x\,dx$

$+\int e^x\sin x\,dx \quad +\int e^x \sin x\,dx$

$e^x\sin x-e^x\cos x = 2\int e^x\sin x\,dx \to \frac{1}{2}[e^x\sin x-e^x\cos x]=\int e^x\sin x\,dx$

$+C$

c. $\int \sin^3\theta\sqrt{\cos\theta}\,d\theta = \int \sin x\,(1-\cos^2 x)\sqrt{\cos x}\,dx$

$\int \sin x(\cos x)^{1/2}-\sin x(\cos x)^{5/2}\,dx$

$u=\cos x$

$du=-\sin x\,dx$

$\int u^{5/2}-u^{1/2}\,du =$

$\frac{2}{7}u^{7/2}-\frac{2}{3}u^{3/2}+C = \frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C$

d. $\int \frac{x}{\left(x^2-6x+10\right)^2}dx$

$\int \frac{x}{[(x-3)^2+1]^2}dx = \int \frac{u+3}{[u^2+1]^2}du = \int \frac{u}{(u^2+1)^2}du + \int \frac{3}{(u^2+1)^2}du$

$u = x-3$
$du = dx$
$x = u+3$

$= \int \frac{1}{2}\frac{1}{v^2}dv$ $\quad v^{-2}$

$v = u^2+1$
$du = 2u\,du$
$\frac{1}{2}dv = u\,du$

$-\frac{1}{2}\frac{1}{v} = \frac{-1}{2}\frac{1}{(u^2+1)}$

$-\frac{1}{2}\frac{1}{x^2-6x+10}$

table of integrals $\quad a=1$, $n=2$

$\int \frac{1}{(a^2+x^2)^n}dx = \frac{1}{2a^2(n-1)}\left[\frac{x}{(a^2+x^2)^{n-1}} + (2n-3)\int \frac{1}{(a^2+x^2)^{n-1}}dx\right]$

$\frac{1}{2(1)}\left[\frac{u}{u^2+1} + (-1)\int \frac{1}{u^2+1}dx\right]$

$-\frac{1}{2}\cdot\frac{1}{x^2-6x+10} + \frac{3}{2}\left[\frac{x-3}{x^2-6x+10} - \arctan(x-3)\right]$

$\frac{1}{2}\left[\frac{3x-10}{x^2-6x+10}\right] - \frac{3}{2}\arctan(x-3) + C$

e. $\int \cosh^3 x \sinh x\, dx$

$u = \cosh x$
$du = \sinh x\, dx$

$\int u^3 du$

$\frac{1}{4}u^4 + C \to \frac{1}{4}\cosh^4 x + C$

f. $\int e^x\sqrt{1-e^{2x}}\,dx$

$u = e^x$
$du = e^x dx$

$\sin\theta = u \quad du = \cos\theta$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\int \sqrt{1-u^2}\,du \qquad \int \cos\theta\cdot\cos\theta\, d\theta = \int \cos^2\theta\, d\theta = \frac{1}{2}\int 1+\cos 2\theta\, d\theta =$

$\frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{1}{2}\left[\arcsin e^x + \frac{1}{2}\cdot 2e^x\sqrt{1-e^{2x}}\right] + C$

16. Determine whether or not the integral converges or diverges. If the integral converges, state its value. (16 points)

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

$\lim_{b\to 1} \arcsin x \Big|_0^b = \lim_{b\to 1} \arcsin b - \cancel{\arcsin 0}^{\,0} =$

$\arcsin 1 = \pi/2$