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Three-Way ANOVA

Three-way ANOVA follows much the same pattern as two-way ANOVA. There are direct effects of each component, each with its own set of hypothesis tests. There are two-way effects that are possible from pairs of treatments, and there is a possible three-way effect from all three factors interacting with each other. Each type of interaction has its set of hypotheses to test, and typically we start with the most complex interaction and work our way down to the primary effects. As we introduce more variables into our model, we can run into other types of issues. Factors may influence each other. One set of factors may prove not to be independent of another factor, so we may see that alone two factors may each prove to have predictive power, but together one or both may end up no longer able to reject the null hypothesis. It is quite rare for higher level interaction terms to survive. As with two-way ANOVA, however, sometimes the interaction term will survive without the main effects term, which may create issues with interpretation of the model (this is uncommon).

As with our previous ANOVA models, we aren't going to compute these by hand, but it's worth seeing how the analysis develops and expands to account to the various effects and interactions.

$$\mu_{ijk} = \mu + \alpha_i + \beta_i + \delta_k + \gamma_{ij}^{AB} + \gamma_{ik}^{AC} + \gamma_{jk}^{BC} + \gamma_{ijk}$$

The notation in our text (Devore) uses gamma for the interaction terms as it did in the two-way model case, leaving delta for the third primary effect (if you know Greek, you know this is out-of-order now). Some sources will use gamma as the third primary effect, and may use multiple letters for the interaction effects going down the line. The subscripts here really tell us what is going on. The primary effects have only one subscript referring to each of the main treatment variables. The combination effects have pairs or triples of subscripts that correspond to the combination of main effects at play. Devore further adds subscripts to clarify which factor variables are involved (when i and j and k are replaced with specific values, this helps to disambiguate). And then the error term ϵ_{ijkl} is the random error on each measurement with mean of zero and standard deviation of σ.

Relevant **sums of squares** are

$$\text{SST} = \sum_i \sum_j \sum_k \sum_l (X_{ijkl} - \overline{X}_{....})^2 \qquad \text{df} = IJKL - 1$$

$$\text{SSA} = \sum_i \sum_j \sum_k \sum_l \hat{\alpha}_i^2 = JKL \sum_i (\overline{X}_{i...} - \overline{X}_{....})^2 \qquad \text{df} = I - 1$$

$$\text{SSAB} = \sum_i \sum_j \sum_k \sum_l (\hat{\gamma}_{ij}^{AB})^2 \qquad \text{df} = (I - 1)(J - 1)$$

$$= KL \sum_i \sum_j (\overline{X}_{ij..} - \overline{X}_{i...} - \overline{X}_{.j..} + \overline{X}_{....})^2$$

$$\text{SSABC} = \sum_i \sum_j \sum_k \sum_l \hat{\gamma}_{ijk}^2 = L \sum_i \sum_j \sum_k \hat{\gamma}_{ijk}^2 \qquad \text{df} = (I - 1)(J - 1)(K - 1)$$

$$\text{SSE} = \sum_i \sum_j \sum_k \sum_l (X_{ijkl} - \overline{X}_{ijk.})^2 \qquad \text{df} = IJK(L - 1)$$

This list doesn't include the SSAC and SSBC sum of squares, but they can be generalized from the SSAB equation.

Null Hypothesis	Test Statistic Value
H_{0A}: all α_i's $= 0$	$f_A = \dfrac{\text{MSA}}{\text{MSE}}$
H_{0AB}: all γ_{ij}^{AB}'s $= 0$	$f_{AB} = \dfrac{\text{MSAB}}{\text{MSE}}$
H_{0ABC}: all γ_{ijk}'s $= 0$	$f_{ABC} = \dfrac{\text{MSABC}}{\text{MSE}}$

Reminder that these are examples. For the full three-way model, we have three main effect hypotheses, three two-way interactions and one three-way interaction: 7 hypotheses to test.

A full ANOVA table for this type of model will have to fill in as follows. These formulas assume the sample size is the same for all components, but recall that we made some adjustments to account for different sample sizes in the various treatment combinations and similar adjustments can be made here. As our models become more complex, it becomes increasingly common to have only one observation in each treatment combination (or less).

SOURCE OF VARIATION	DF	SUM OF SQUARES	MEAN SQUARE	F
A	$I-1$	SSA	MSA	F_A
B	$J-1$	SSB	MSB	F_B
C	$K-1$	SSC	MSC	F_C
AB	$(I-1)(J-1)$	$SSAB$	$MSAB$	F_{AB}
AC	$(I-1)(K-1)$	$SSAC$	$MSAC$	F_{AC}
BC	$(J-1)(K-1)$	$SSBC$	$MSBC$	F_{BC}
ABC	$(I-1)(J-1)(K-1)$	$SSABC$	$MSABC$	F_{ABC}
ERROR	$IJK(L-1)$	SSE	MSE	
TOTAL	$IJKL-1$	SST		

An example of a three-way model in R using the mtcars data set using the factors gear, am and carb to predict mpg.

```
            Df Sum Sq Mean Sq F value   Pr(>F)    
gear         2  483.2  241.62  26.382 1.87e-06 ***
carb         5  425.1   85.02   9.283 8.80e-05 ***
am           1   14.1   14.08   1.538    0.229    
carb:am      2   11.3    5.64   0.616    0.550    
Residuals   21  192.3    9.16                     
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

However, if we remove the interaction term, all the variable become signficant.

```
              Df Sum Sq Mean Sq F value   Pr(>F)    
gear           2  483.2  241.62  27.294 8.46e-07 ***
am             1   72.8   72.80   8.224 0.008699 ** 
carb           5  366.4   73.28   8.278 0.000136 ***
Residuals     23  203.6    8.85                     
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

As we noted, the full model of all possible interactions is quite complex, and we would need a complete layout (at least one observation for all conditions) in order to detect those effects which often do not rise to the level of significance. We can limit our model to just main effects and then we don't have to collect quite as much data. We can carefully chose the combinations of factors to measure. These are called incomplete layouts. They don't have the power to test interactions, but they are more efficient, especially as we continue to add more variables with more treatment levels.

The Latin Squares design is the most popular incomplete layout design. It depends on all the factor variables having the same number of levels.

A \ B (C)	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

A \ B (C)	1	2	3	4
1	3	4	2	1
2	4	2	1	3
3	2	1	3	4
4	1	3	4	2

A \ B (C)	1	2	3	4	5
1	4	3	5	2	1
2	3	1	4	5	2
3	1	5	2	3	4
4	5	2	1	4	3
5	2	4	3	1	5

The idea is somewhat like sudoku. Each row and each column should have one observation for the third factor in it. Like sudoku, the number of combinations increases as the number of levels increases (there are 12 3×3 Latin squares). This means that the number of observations extends like N^2 instead of N^3. For 5 levels, you'd need 25 observations, rather than 125.

Because the design is very regular, and we don't have to worry about interactions, the model calculations simplify.

Sums of squares for a Latin square experiment are

$$\text{SST} = \sum_i \sum_j (X_{ij(k)} - \overline{X}_{...})^2 \qquad \text{df} = N^2 - 1$$

$$\text{SSA} = \sum_i \sum_j (\overline{X}_{i..} - \overline{X}_{...})^2 \qquad \text{df} = N = 1$$

$$\text{SSB} = \sum_i \sum_j (X_{.j.} - \overline{X}_{...})^2 \qquad \text{df} = N - 1$$

$$\text{SSC} = \sum_i \sum_j (\overline{X}_{..k} - \overline{X}_{...})^2 \qquad \text{df} = N - 1$$

$$\text{SSE} = \sum_i \sum_j [X_{ij(k)} - (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_k)]^2 \qquad \text{df} = N - 1$$

$$= \sum_i \sum_j (X_{ij(k)} - \overline{X}_{i..} - \overline{X}_{.j.} - \overline{X}_{..k} + 2\overline{X}_{...})^2 \qquad \text{df} = (N-1)(N-2)$$

$$\text{SST} = \text{SSA} + \text{SSB} + \text{SSC} + \text{SSE}$$

And we can apply Tukey's method to see where the differences in the means are and how they might be grouped if at all.

$$w = Q_{\alpha,N,(N-1)(N-2)} \cdot \sqrt{\text{MSE}/N}.$$

While a Latin square design is extremely common, there are alternatives, such as a Greco-Roman design model. Linked in the references is an article on Latin square models and other alternatives.

Review for Exam #2

Main topics to be covered on the second exam:

- Sampling distributions
- Confidence intervals
- Maximum Likelihood Estimates
- Method of Moments
- Hypothesis tests
  - One- and two-sample tests
  - One- and two-sided tests
  - Means and proportions
  - ANOVA (using technology)
  - Tukey's method (using technology)
  - Power
  - Significance level
  - Other parametric tests?

As with the last exam, students will be given data to take home and analyze ahead of the test. In class, students will be asked questions about their analysis. In-class questions can be done with a calculator, with analysis from R that I provide, or explaining concepts. The focus will be on material covered since Exam #1.

Next class: Exam #2

References:

1. https://faculty.ksu.edu.sa/sites/default/files/probability_and_statistics_for_engineering_and_the_sciences.pdf
2. https://online.stat.psu.edu/stat503/lesson/4/4.3
3. https://www.jstor.org/stable/2528324
4. https://math.unm.edu/~fletcher/SUPER/chap17.pdf