9/23/2021

Determinants

Determinants are a single number that represents a matrix. Non-zero values mean the matrix is invertible, and determinants equal to zero mean it is not invertible.

Not invertible = singular
Invertible = nonsingular

2x2 matrix

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Co-factor method for finding determinants
The basic idea is to expand on a row or column of the matrix and reduce the size of the submatrices that we need to calculate determinants for.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

First entry times the rest of the matrix with the row and column deleted, then flip sign and continue…

$$1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$$

$$1(2-4) - 2(6-2) + 4(12-2) = -2 - 8 + 40 = 30$$

This was using the first row. What if we use the second column?

The rule for signs is $(-1)^{i+j}$

$$-2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$$

$$-2(6-2) + 1(2-8) - 4(1-12) = -8 - 6 + 44 = 30$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}\begin{matrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{matrix}$$

$$(1)(1)(2) + 2(1)(2) + 4(3)(4) - 2(1)(4) - 4(1)(1) - 2(3)(2) =$$
$$2 + 4 + 48 - 8 - 4 - 12 = 54 - 24 = 30$$

This process will work for 3x3s, ***<u>but it does not work for matrices of any other size!!!!</u>***
4x4s

$$\begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{bmatrix}$$

Because of the zeros, we are going expand on Row 2 to take advantage of them.

$$-0\left|\quad\right| + 0\left|\quad\right| - 3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix} + 0\left|\quad\right|$$

$$-3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

$$-3\left[2\begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix} - 0\left|\quad\right| + 5\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}\right]$$

$$-3[2(-10+8) + 5(-4+4)] = -6(-2) = 12$$

Properties of Determinants
Determinants are only defined for square matrices
A matrix with a zero determinant is not invertible

$$\det(A^T) = \det(A)$$

$$\det(AB) = \det(A)\det(B) = \det(BA)$$

Row operations on matrices and their relationship to determinants
$R_1 \leftrightarrow R_2$: the effect is to change the sign of the determinant
$kR_1 + R_2 \to R_2$: effect is to do nothing at all
$kR_1 \to R_1$: effect is to change the determinant by k

If you multiply a matrix by a constant, the effect is k^n

$$\begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{bmatrix}$$

If you can get the matrix into echelon (triangular) form, then the determinant is just the product of the diagonals.

$-3R_1 + R_3 \to R_3$: no change
$6R_1 + R_4 \to R_4$: no change

$$\begin{bmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & 10 & -8 & -2 \\ 0 & 35 & 29 & 6 \end{bmatrix}$$

$$1\begin{vmatrix} -2 & -4 & 0 \\ 10 & -8 & -2 \\ 35 & 29 & 6 \end{vmatrix}$$

$3R_2 + R_3 \rightarrow R_3$: no change

$$1\begin{vmatrix} -2 & -4 & 0 \\ 10 & -8 & -2 \\ 65 & 5 & 0 \end{vmatrix}$$

$$1\left[-(-2)\begin{vmatrix} -2 & -4 \\ 65 & 5 \end{vmatrix}\right] = 1(2)(-10+260) = 2(250) = 500$$

Row reduce (feel free to be clever to save work), and use cofactor expansion to reduce the size of the matrix.

Cramer's Rule

Solution method for systems of equations, it uses determinants, it only works when the system is independent.

A is the coefficient matrix.
A_i is the matrix where ith column is replaced by the b constant vector from the system

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$$\begin{bmatrix} 4 & 1 & 6 \\ 3 & 2 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, \det(A) = 8 - 3 = 5$$

$$A_1 = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}, \det(A_1) = 12 - 7 = 5$$

$$A_2 = \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}, \det(A_2) = 28 - 18 = 10$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{5}{5} = 1$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{10}{5} = 2$$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ the equivalent reduced augmented matrix

Area and Volume
If you have a parallelogram defined by two vectors, then the determinant of the matrix that contains the two vectors is the area of the parallelogram.

If you have a parallelepiped defined by three vectors, then the determinant of the matrix of those vectors is the volume of the parallelepiped. (equivalent to the triple scalar product).

End Chapter 3.

Review for Exam #1.

Archive Site: https://betsymccall.net/prof/courses/fall21/nova/linear.html