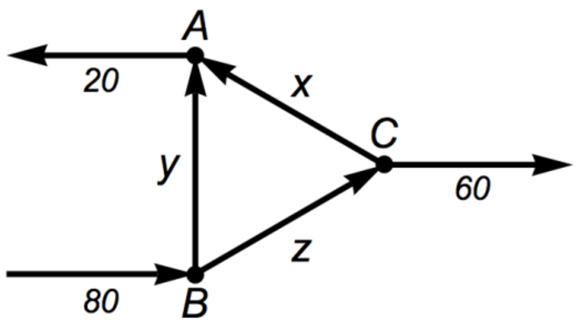
9/16/2021

Applications, Inverses, Determinants(?)

Traffic Control Problem



$$x + y = 20$$
$$80 = y + z$$
$$z = x + 60$$

$$\begin{cases} x + y = 20 \\ y + z = 80 \\ -x + z = 60 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 80 \\ -1 & 0 & 1 & 60 \end{bmatrix}$$

Solve.

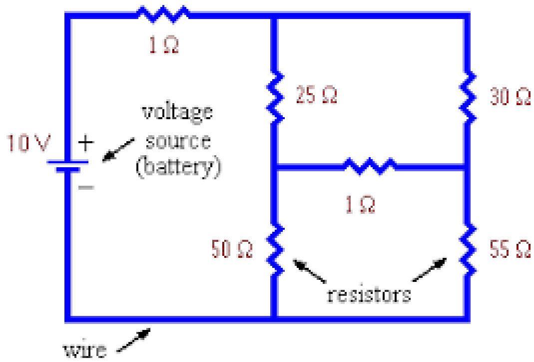
$$\begin{bmatrix} 1 & 0 & -1 & -60 \\ 0 & 1 & 1 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - z = -60$$
$$y + z = 80$$

Negative values of a variable reverse the flow (the traffic flows in the opposite direction of the arrow on the graph). Positive goes with the arrow. If the system is dependent, then you have to choose the free variable to obtain the traffic pattern.

Circuit Problem. $V = IR$

Set up (in the case) three equations of the loop currents I_1, I_2, I_3 to solve for each of the currents.



$$10 = 50I_1 - 50I_3 + 25I_1 - 25I_2 + I_1$$
$$0 = 25I_2 - 25I_1 + I_2 - I_3 + 30I_2$$
$$0 = I_3 - I_2 + 50I_3 - 50I_1 + 55I_3$$

$$76I_1 - 25I_2 - 50I_3 = 10$$
$$-25I_1 + 56I_2 - I_3 = 0$$
$$-50I_1 - I_2 + 106I_3 = 0$$

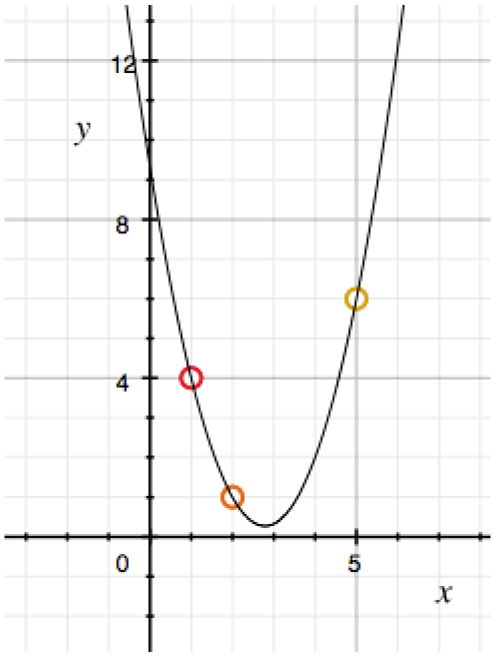
Solve in a matrix.

$$I_1 = 0.244934\ldots, I_2 = 0.111427\ldots, I_3 = 0.116586\ldots$$

Interpolating polynomial problems.

Interpolate a quadratic polynomial (degree 2) to pass through these three points.
General quadratic equation is $ax^2 + bx + c = y$

Red point (1,4): $a(1)^2 + b(1) + c = 4 \;\; \rightarrow a + b + c = 4$
Orange point (2,1): $a(2)^2 + b(2) + c = 1 \;\; \rightarrow 4a + 2b + c = 1$
Yellow point (5,6): $a(5)^2 + b(5) + c = 6 \;\; \rightarrow 25a + 5b + c = 6$



$$a = 1.16666\ldots, b = -6.5, c = 9.33333\ldots.$$

$$a = \frac{7}{6}, b = -\frac{13}{2}, c = \frac{28}{3}$$

$$y = \frac{7}{6}x^2 - \frac{13}{2}x + \frac{28}{3}$$

Inverses.

$$AA^{-1} = I$$
$$A^{-1}A = I$$

A^{-1} this is the inverse of A (or A-inverse).
For any matrix to have an inverse, it must first be square (or the above relations are impossible). But not all matrices have inverses.
For 2x2 matrices, there is an actual formula for the inverse. For larger matrices, there is a computation process where we can obtain the inverse.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse will exist as long as $ad - bc \neq 0$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7 - 8}\begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1(-7) + 2(4) & 1(2) + 2(-1) \\ 4(-7) + 4(7) & 4(2) + 7(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -7(1) + 2(4) & -7(2) + 2(7) \\ 4(1) - 1(4) & 4(2) - 1(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Ax = b$$

Like:

$$mx = b$$

$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Check with inverses.

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1}b = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3(-1) + 2(3) \\ 1(-1) + 1(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Properties of inverses:

$$(A^T)^T = A, (A^{-1})^{-1} = A$$
$$(AB)^T = B^T A^T, (AB)^{-1} = B^{-1}A^{-1}$$

$$AB(AB)^{-1} = ABB^{-1}A^{-1} = A(I)A^{-1} = AA^{-1} = I$$

$$(A^{-1})^T = (A^T)^{-1}$$

Elementary matrices, E_i
A matrix that has the effect of a row operation.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 + R_3 \to R_3$$

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & 6 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

If you multiply A by E_1 you obtain A_1

$$E_1 A = A_1$$

$$R_1 \to R_2$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(E_n \cdot \ldots E_2 E_1)A = I$$
$$(E_n \cdot \ldots E_2 E_1) = A^{-1}$$

Because of this, there is a row operation process to find the inverse

$$[A \quad I] \to [I \quad A^{-1}]$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix}$$

$$R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$2R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Invertible matrix theorem.
A is a square matrix. The following statements are equivalent:

1. A is invertible (A has an inverse)
2. A is row equivalent to the identity
3. A has n pivots
4. The equation $Ax = 0$ has only the trivial solution
5. The columns of A are linearly independent
6. The linear transformation $x \mapsto Ax$ is one-to-one.
7. The equation $Ax = b$ has at least one solution for each b in R^n
8. The columns of A span R^n
9. The linear transformation $x \mapsto Ax$ maps R^n onto R^n

10. There is an $n \times n$ matrix C such that $CA = I$
11. There is an $n \times n$ matrix D such that $AD = I$
12. A^T is invertible.

If one is true, all the remaining 11 are true. If one is false, then all the remaining 11 are false.

We'll do determinants next week.