11/4/2021

Eigenvalues/Eigenvectors/Eigenpairs
Review for Exam

Eigen = "own", characteristic property

Eigenvalue = is a constant/scalar that has a special property associated with a matrix
Eigenvector = is a vector that has a special/characteristic property associated with a matrix.

They satisfy a particular equation:

$$Ax = \lambda x$$

In the direction of a particular vector, the matrix acts like a scalar.

Example.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$
$$x = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Is this x an eigenvector for A?

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6(1) + 6(-5) \\ 5(6) + 2(-5) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

X is an eigenvector for A, and the corresponding eigenvalue is -4.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -24 \\ 20 \end{bmatrix} = \begin{bmatrix} 1(-24) + 6(20) \\ 5(-24) + 2(20) \end{bmatrix} = \begin{bmatrix} 96 \\ -80 \end{bmatrix} = 16 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4 \begin{bmatrix} -24 \\ 20 \end{bmatrix}$$

When it's not:

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 6(-2) \\ 5(3) + 2(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

Not a multiple of the original vector
So this second vector is not an eigenvector

Suppose you want to test for a given λ whether it is an eigenvalue or not.

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0$$

Solve $A - \lambda I$ for the nullspace (what vector is the solution to this homogeneous equation)?

(the eigenvector must be non-zero)

If you get the trivial solution, it's because λ is not eigenvalue (or there is an arithmetic mistake).

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \lambda = 7$$

Find the eigenvector that corresponds to the eigenvalue $\lambda = 7$.

$$(A - 7I)x = 0$$
$$(A - \lambda I) = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$$5x_1 - 5x_2 = 0$$
$$5x_1 = 5x_2$$
$$x_1 = x_2$$
$$x_2 = x_2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

This is the eigenvector that goes with the eigenvalue of 7.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 6(1) \\ 5(1) + 2(1) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = -4, x_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$
$$\lambda_2 = 7, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If the matrix is 3x3 or larger, row reduce the resulting $A - \lambda I$ matrix in your calculator to reduced echelon form to find the nullspace (vector).

The Characteristic equation.

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors for A.

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 6 \\ 5 & 2 - \lambda \end{bmatrix}$$

What do we know about matrices that are dependent?
Find the determinant and set it equal to zero.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 30 = 0$$
$$2 - \lambda - 2\lambda + \lambda^2 - 30 = 0$$

$$\lambda^2 - 3\lambda - 28 = 0$$

This is the characteristic equation.
The solutions to this equation will get us the eigenvalues.

$$(\lambda + 4)(\lambda - 7) = 0$$

$$\lambda = -4, 7$$

Check like we did before when we had the eigenvalues.

$$A - \lambda I = \begin{bmatrix} 1-(-4) & 6 \\ 5 & 2-(-4) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

$$5x_1 + 6x_2 = 0$$

$$x_1 = -\frac{6}{5}x_2$$
$$x_2 = x_2$$

$$x = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

Eigenvectors generally are preferred to be in terms of small integers, rather than decimals or fractions (or large numbers) whenever possible. (called normalizing)

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

What is the determinant of this matrix?

$$(-4)(7) = -28$$

The product of the eigenvalues is equal to the determinant.

You can’t have more eigenpairs than the number of dimensions of the matrix (n)

Invertible matrix theorem:
s. The number 0 is not an eigenvalue of the matrix.
t. the determinant of A is not zero

The eigenvalues of A^T are the same as A. The eigenvectors will be different.

Similarity/Similar Matrices

A matrix A is similar to matrix B if there exists a matrix P such that:

$$P^{-1}AP = B$$
$$AP = PB$$
$$A = PBP^{-1}$$

$$det(A) = det(B)$$

Even if a matrix has the same eigenvalues (and determinant) as another matrix, they are not guaranteed to be similar.

Generally, P is based on the eigenvectors of one of the matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Exam Review

Coordinate Transformations (handout covers all three cases)

$$P_B[x]_B = x$$

$$P_B[x]_B = P_C[x]_C$$

Orthogonal case: (not on the exam)

$$c_i = \frac{x \cdot b_i}{b_i \cdot b_i}$$

Basis is a set of vectors that produce a unique representation of every "point" in the set/space.
-independent
-span the space

Number of vectors in the basis is the dimension of the space

Covers chapters 3 and 4.

Determinants – be able to find the determinant for large matrices by cofactor, and by the row-reducing method (in this method, you can only use the cofactor method after reducing one column (or row)).

Applications of determinants: Cramer's rule, applications to area and volume

Properties of determinants – (a lot of these were on the second proof set)

Vector spaces – proofs or counterexamples

Dimensions of subspaces (for linear transformation)

Coordinate transformations

Nullspace/bases for nullspaces, column spaces, etc.

Bases: verifying independence and span