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Diagonalization, Complex Eigenvalues, Applications

Diagonal matrices are matrices with non-zero components only on the diagonal.

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

For a diagonal matrix, the standard basis vectors are the eigenvectors, and the diagonal elements are the corresponding eigenvalues.

We want to be able to find a similarity transformation that converts a general matrix into a diagonal one.

If an nxn matrix is diagonalizable, then it will have n linearly independent eigenvectors.

If it is diagonalizable, then $A = PDP^{-1}$, where P is the similarity transformation, and P is the matrix of the (independent) eigenvectors of A. And the diagonal entries of D will be the eigenvalues of A.

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

$$(7 - \lambda)(1 - \lambda) + 8 = 0$$

$$\lambda^2 - 8\lambda + 7 + 8 = \lambda^2 - 8\lambda + 15 = 0$$
$$(\lambda - 3)(\lambda - 5) = 0$$
$$\lambda = 3,5$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\lambda_1 = 3$$
$$\begin{bmatrix} 7-3 & 2 \\ -4 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}$$

$$4x_1 + 2x_2 = 0$$
$$x_1 = -\frac{1}{2}x_2$$
$$x_2 = x_2$$

$$v = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 5$$

$$\begin{bmatrix} 7-5 & 2 \\ -4 & 1-5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$
$$x_1 = -x_2$$
$$x_2 = x_2$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$A = PDP^{-1}$$
$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

What is A^4?

$$A = PDP^{-1}$$

$$A^4 = PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1}$$
$$= PD^4P^{-1}$$
$$\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3^4 & 0 \\ 0 & 5^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$$

How do we know if a matrix has n independent eigenvectors (to be diagonalizable)?

A matrix is diagonalizable if there are n distinct eigenvalues. (Because every vector from a eigenvalue is independent of vectors for different eigenvalues.)

Repeated eigenvalues MAY produce more than one independent eigenvector (eigenspace=subspace defined by the eigenvectors for a particular eigenvalue), the eigenvalue might have the same dimension as the number of repetitions of the factor in the characteristic equation or it might not. There is no way to know without looking for the nullspace. It might produce fewer vectors.

You are guaranteed only one vector for each eigenvalue.

3x3 matrix, with two eigenvalues. Each eigenspace is one-dimensional. Is this matrix diagonalizable? No.

3x3 matrix with two eigenvalues, and one eigenspace is one-dimensional and one is two-dimensional. Is this matrix diagonalizable? Yes.

Consider $\lambda(\lambda - 2)^2 = 0$ is the characteristic equation.

Complex Eigenvalues

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$(1 - \lambda)(3 - \lambda) + 2 = 0$$

$$\lambda^2 - 4\lambda + 3 + 2 = \lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$A - \lambda I = \begin{bmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{bmatrix} = \begin{bmatrix} 1-2-i & -2 \\ 1 & 3-2-i \end{bmatrix} = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}$$

$$x_1 + (1-i)x_2 = 0$$
$$x_1 = (-1+i)x_2$$
$$x_2 = x_2$$

$$v_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$$A - \lambda I = \begin{bmatrix} 1-(2-i) & -2 \\ 1 & 3-(2-i) \end{bmatrix} = \begin{bmatrix} 1-2+i & -2 \\ 1 & 3-2+i \end{bmatrix} = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$$

$$x_1 + (1+i)x_2 = 0$$
$$x_1 = (-1-i)x_2$$
$$x_2 = x_2$$

$$v_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$$

This trick with the complex conjugates also works for real solutions with square roots.

Complex eigenvalues (eigenvectors) mean that the matrix is acting like a scaled rotation matrix.

Rotation matrices: $T = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

If our 2x2 matrix has complex eigenvalues, for $\lambda = a - bi$ ($b \neq 0$), and the corresponding eigenvector, there is a similarity transformation P so that $A = PCP^{-1}$, where $P = [Re(v) \;\; Im(v)]$, and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

For the previous example:

$$P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is a scaled rotation matrix. $\cos\theta = \frac{a}{r}, \sin\theta = \frac{b}{r}, r = \sqrt{a^2 + b^2} = \sqrt{5}$

$$C = \sqrt{5}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = 0.46367609 \ldots radians = 26.57°$$

Applications.

Markov Chains
Discrete Dynamical Systems
Systems of Ordinary Differential Equations.

Markov chains are matrices that represent probabilities of transitioning between states.
If the columns of the matrix add up to 1 (kind of normalization) then they can represent probabilities.

Markov chains tend (in most cases) toward a steady state. $\lim_{n\to\infty} P^n x_0 = q$ or $Pq = q$.

$Pq = q$ vs. $Ax = \lambda x$

$\lambda = 1$ is always an eigenvalue of the matrix P. it is the eigenvalue that corresponds to the steady state vector.

$$Pq - q = (P - I)q = 0 \;\; or \; (A - \lambda I)x = 0$$

Discrete Dynamical systems are systems that change over time, but are only measured at discrete times.

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.7 & 1.1 \end{bmatrix}$$

The behavior of the system will depend on the eigenvalue (and the eigenvectors).

If the eigenvalue is bigger than 1, then both populations will grow. If both eigenvalues are less than 1, then both populations will die out. If one eigenvalue is 1, then there is a steady state.

Use the eigenvectors to make a basis for the space: $x_0 = c_1 v_1 + c_2 v_2$

$$x_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$$

Zero population is the baseline. If the population is tending toward 0 vector, we call this an attractor. If the population tends to grow (in both directions), then the 0 vector is a repeller. 0 can be a saddle point, if one eigenvalue is bigger than 1, and one is less than 1. (assuming that all eigenvalues are positive).

In theory, you can have complex eigenvalues (that produce a rotation), but negative values of the state vector have no meaning. – if 0 is really a zero. If the eigenvalue is complex, it's the magnitude of the discrete eigenvalue that determines attraction or repelling.

Systems of ordinary differential equations.

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$x' = Ax$$

$$x = e^{\lambda t}$$

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$x = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} e^{\lambda_2 t}$$

The λs are the eigenvalues of the A matrix, and the c-vectors are the eigenvectors.

If λ is real and positive, the solution grows away from the origin. If they are real and negative, they grow toward the origin. If the there is a $\lambda = 0$, this is a steady state. If they are opposite signs, then the origin is a saddle point. If they are complex, then there is a rotation. The real part of the eigenvalue will determine if the origin attracts or repels.