10/21/2021

HW Q.

Is the set of polynomials $\{1, t+2, t^2+3t-6\}$ as basis for P_2?

Rewrite the polynomials as vectors. Generally, the constants are the first coefficient, and then increasing degree as the vector gets longer.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-6\\3\\1\end{bmatrix}\right\}$$

Row reduce this to look for pivots: it's already in echelon form. There is one pivot in each column (independence) and one pivot in each row (span), it is a basis for P_2.

Inner Products (chapter 6)

In the vector world of R^n, the inner product is also called the dot product or the scalar product. $u \cdot v = \langle u|v\rangle$.

The dot product is defined in R^n as $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

Ex. $u = \begin{bmatrix}-2\\3\\5\end{bmatrix}, v = \begin{bmatrix}1\\-2\\4\end{bmatrix}, u \cdot v = (-2)(1) + (3)(-2) + (5)(4) = -2 - 6 + 20 = 12$

All inner products must satisfy:

$$u \cdot v = v \cdot u \qquad \langle u|v\rangle = \langle v|u\rangle$$

$$(u+v) \cdot w = u \cdot w + v \cdot w \qquad \langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$$

$$(cu) \cdot v = c(u \cdot v) = u \cdot (cv) \qquad \langle cu|v\rangle = c\langle u|v\rangle = \langle u|cv\rangle$$

$u \cdot u \geq 0$ and only $u \cdot u = 0$ iff $u = 0$ $\qquad \langle u|u\rangle \geq 0$

Sometimes the vector definition of a dot product can be written as $u \cdot v = u^T v$

$$u^T = \begin{bmatrix}-2 & 3 & 5\end{bmatrix}$$

$$u \cdot v = u^T v = \begin{bmatrix}-2 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\-2\\4\end{bmatrix} = (-2)(1) + 3(-2) + 5(4) = 12$$

This is a 1x1 matrix, or a scalar.

Length/norm/magnitude of a vector: $\|v\| = |v| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} = \sqrt{v \cdot v}$

$$\|v\|^2 = v \cdot v$$

$$\|cv\| = |c|\|v\|$$

Unit vector: a vector with length = 1

$$\hat{v} = \frac{v}{\|v\|}$$

Suppose I wanted a unit vector in the direction of $v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. $\hat{v} = \frac{<1,-2,2>}{\sqrt{1^2+(-2)^2+2^2}} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

You can use unit vectors as direction pointers that don't contribute any length to the problem.

The distance between two points (vectors).

$$dist(u, v) = \|u - v\|$$

Orthogonal vectors. (in geometry, orthogonal = perpendicular)
Two vectors are orthogonal when $u \cdot v = 0$.

$$u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$u \cdot v = (1)(2) + (-2)(1) = 2 - 2 = 0$$

Think about a set (subspace) of vectors W. The set of vectors z that are orthogonal to every vector in W is in the set $W^{\perp}$ (W-perp).

$\dim(W) + \dim(W^{\perp}) = \dim(V)$ (whole vector space)

The zero vector is the only vector in both W and $W^{\perp}$.

$$(Row\ A)^{\perp} = Nul\ A$$
$$(Col\ A)^{\perp} = Nul\ (A^T)$$

Angles and Dot Products

$$u \cdot v = \|u\|\|v\| \cos\theta$$

θ is the angle between the vectors.

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

Inner Product Space = a vector space that has a defined inner product. The inner product still has to satisfy the four properties listed earlier.

Function spaces (sets of functions that have a basis set, and satisfy the definitions of a vector), we define an inner product on the set of function that satisfies all these properties.

$$\langle f|g\rangle = \int_a^b f(x)g(x)dx$$

$$f(x) = 1 + x$$
$$g(x) = x^2$$

$$\langle f|g\rangle = \int_0^1 f(x)g(x)dx$$

What is $\|f\|$?

$$\langle f|f\rangle = \int_0^1 f(x)f(x)dx = \int_0^1 (1+x)^2 dx = \int_0^1 1 + 2x + x^2 dx = x + x^2 + \frac{1}{3}x^3|_0^1 = 1 + 1 + \frac{1}{3} = \frac{7}{3}$$

$$\|f\| = \sqrt{\langle f|f\rangle}$$

A unit vector in the "direction of f" $\hat{f} = \frac{f}{\|f\|} = \left(\sqrt{\frac{3}{7}}\right)(1+x)$

What is the distance between f and g?

$$\|f - g\| = \sqrt{\langle f - g|f - g\rangle} = \sqrt{\int_0^1 (1 + x - x^2)^2 dx}$$

Finish integrating.

What is the angle between f and g? (are f and g orthogonal?)

$$\langle f|g\rangle = \int_0^1 (1+x)x^2 dx = \int_0^1 x^2 + x^3\, dx = \frac{1}{3}x^3 + \frac{1}{4}x^4|_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

They are not orthogonal. The "angle" between the functions is acute (inner product is positive).

$$\cos(\theta) = \frac{\langle f|g\rangle}{\|f\|\|g\|} = \frac{\left(\frac{7}{12}\right)}{\sqrt{7/3}\|g\|}$$

Orthogonal set is a set of vectors that are all mutually perpendicular/orthogonal to each other. Orthogonality implies independence.

Orthogonal basis – it is a basis where all the vectors are also mutually orthogonal.

1) Independence
2) Spans the space
3) Mutually orthogonal

Orthonormal basis – is a basis where the vectors are all orthogonal and have a length of 1

1) Independent
2) Span
3) Mutually orthogonal
4) Length of 1 (norm)

Projection (orthogonal projection)

$$y_{\parallel} = proj_v y = \left(\frac{y \cdot v}{v \cdot v}\right) v = \left(\frac{y \cdot v}{\|v\|^2}\right) v$$

Project $y = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ in the direction of $v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

What is the orthogonal project of y in the direction of v, or y onto v?

$$proj_v y = \frac{1(1) + 2(3) + 4(-2)}{1^2 + 3^2 + (-2)^2} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = -\frac{1}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{14} \\ -\frac{3}{14} \\ \frac{1}{7} \end{bmatrix} = y_{\parallel}$$

What if I want to decompose y into $y_{\parallel}$ (the portion parallel to v), and the portion perpendicular to v? ($y_{\perp}$)

$$y = y_{\parallel} + y_{\perp}$$

$$y_{\perp} = y - y_{\parallel} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -\frac{1}{14} \\ -\frac{3}{14} \\ \frac{1}{7} \end{bmatrix} = \begin{bmatrix} \frac{15}{14} \\ \frac{31}{14} \\ \frac{27}{7} \end{bmatrix}$$

If we are projecting onto a vector space (subspace), if the vectors that define the space are not orthogonal, then our projections onto each basis vector "overlap" and so don't have a handy way shortening the calculations for projections. But, if the vectors are orthogonal for the basis, then we can project onto the subspace one basis vector at a time, and then add up the results.

Coordinate transformations from a vector into the orthogonal basis.

The weights (coordinates) for each of the basis vector (when the basis is orthogonal) is just the projection (orthogonal projection) onto that basis vector.

$$y = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

$$c_i = \frac{y \cdot b_i}{b_i \cdot b_i}$$

U is a matrix with columns that are an orthonormal basis, then: $U^T U = I$

There is a new link for the Tuesday office hours. Use the new one, not the old one.