Math 266, Exam #3, Fall 2021 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. You must show all hand-written work on this part of the exam. Answers with no work will receive only 1 point. When you are finished with this portion of exam, continue with Part II.

1. For each of the matrices shown below, find the eigenvalues and eigenvectors of the matrix. (10 points each)
   a. $A = \begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$

$(-4-\lambda)(7-\lambda) - 12 = 0$

$-28 + 4\lambda - 7\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 40 = 0$

$(\lambda - 8)(\lambda + 5) = 0$

$\lambda = 8, \ \lambda = -5$

$\begin{bmatrix} 1 & 2 \\ 6 & 12 \end{bmatrix}$ $\lambda_2 = -5$

$x_1 = -2x_1$

$x_2 = x_2$

$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -12 & 2 \\ 6 & -1 \end{bmatrix}$ $\lambda_1 = 8$

$6x_1 = x_2$

$x_1 = \frac{1}{6}x_2$

$x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

   b. $B = \begin{bmatrix} -8 & -1 \\ 7 & 4 \end{bmatrix}$

$(-8-\lambda)(4-\lambda) + 7 = 0$

$-32 + 8\lambda - 4\lambda + \lambda^2 + 7 = 0$

$\lambda^2 + 4\lambda - 25 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16 + 100}}{2} = -2 \pm \sqrt{29}$

$\lambda_1 = -2 + \sqrt{29}$

$\begin{bmatrix} -6-\sqrt{29} & -1 \\ 7 & 6-\sqrt{29} \end{bmatrix}$ $v_1 = \begin{bmatrix} -6+\sqrt{29} \\ 7 \end{bmatrix}$

$7x_1 = (-6+\sqrt{29})x_2$

$x_1 = \dfrac{-6+\sqrt{29}}{7}x_2$

$x_2 = x_2$

$\lambda_2 = -2 - \sqrt{29}$

$v_2 = \begin{bmatrix} -6-\sqrt{29} \\ 7 \end{bmatrix}$

2. The weather in a particular town in the Pacific northwest is classified as either good, indifferent or bad. If the weather is good on a particular day, the probability it will remain good the next day is only 11%, and there is a 45% chance it will be indifferent. If the weather is indifferent on a particular day, there is a 20% chance it will become good, and a 42% chance it will become bad. If the weather is bad on a particular day, there is a 48% chance it will be bad the next day, and a 21% chance it will become indifferent. Construct the stochastic matrix that models this problem. (8 points)

$$\begin{bmatrix} .11 & .20 & .31 \\ .45 & .38 & .21 \\ .44 & .21 & .48 \end{bmatrix}$$

3. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (3 points each)

a. T F If λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3.

b. T F If zero is not an eigenvalue of A, then the determinant of A is non-zero.

c. T F If $\vec{v}$ is an eigenvector of A, then $\vec{v}$ is also an eigenvector of e^A.

d. T F If A is diagonalizable, then A is invertible.

e. T F Row operations on a matrix do not change its eigenvalues.

f. T F Similar matrices always have the same eigenvalues.

g. T F The dimension of the eigenspace of an $n \times n$ matrix is always n.

h. T F The vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} 7 & -7 \\ 6 & 4 \end{bmatrix}$.

i. T F A 7x7 matrix has three eigenvalues, one eigenspace is one dimensional, one eigenspace is two dimensional, and one eigenspace is three dimensional; therefore this matrix is diagonalizable.

j. T F Stochastic matrices, regardless of their size, always have at least one real eigenvector.

k. T F The real eigenvalues of a discrete dynamical system must always both attract or repel from the origin.

l. T F In a system of ODEs, the magnitude of λ determines whether a complex eigenvalues causes the origin to repel or attract.

m. T F The dot product or scalar product is one type of inner product.

n. T F To be an inner product space, a space needs to be both vector space, and have a particular inner product defined on it.

o. T F Normalizing a vector refers to making a vector pointing in a particular direction have components that satisfy certain conditions.

p. T F The distance between two points (vectors) in R^n is defined to be $||\vec{u}|| - ||\vec{v}||$.

4. Find the equilibrium vector of the matrix $P = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}$ algebraically. Be sure to properly normalize the vector. (8 points)

$$P - I = \begin{bmatrix} -.9 & .6 \\ .9 & -.6 \end{bmatrix}$$

$$-.9x_1 = -.6x_2$$

$$x_1 = \frac{2}{3}x_2$$

$$x_1 = x_2$$

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

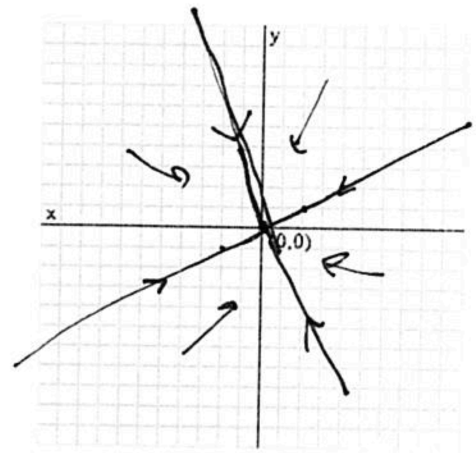
$$\vec{x} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

5. For each of the situations below, determine the properties of the **discrete dynamical system**. Is the origin an attractor, a repeller, or a saddle point? Sketch the eigenvalues on the graphs provided (if they are real) and plot some sample trajectories. (5 points each)

   a. $\lambda_1 = 0.9, \lambda_2 = 0.4, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

attractor

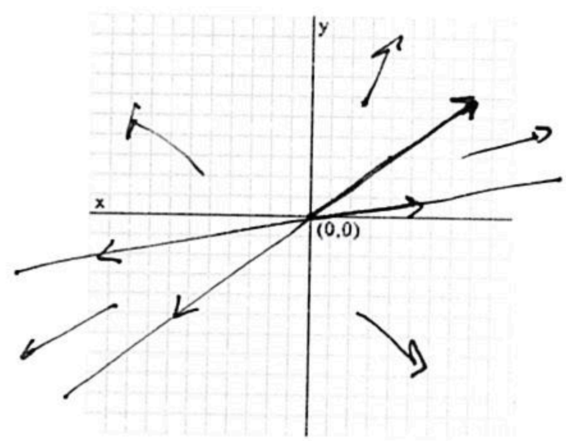
both $|\lambda| < 1$



   b. $\lambda_1 = 1.2, \lambda_2 = -3, \vec{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

repeller

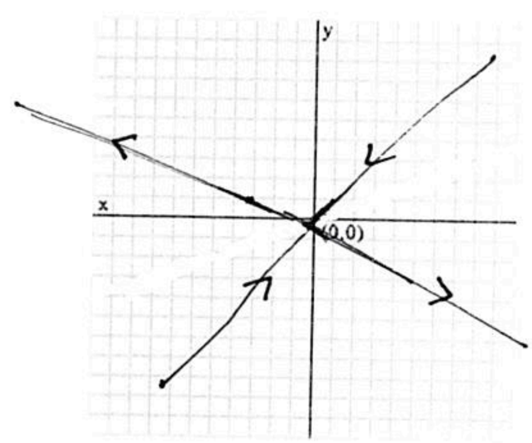
both $|\lambda| > 1$



   c. $\lambda_1 = -2.1, \lambda_2 = 0.4, \vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Saddle point

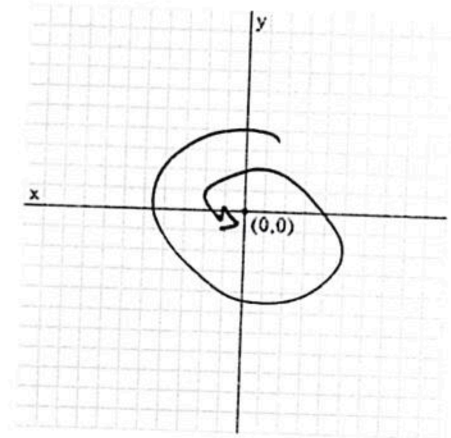
one $|\lambda| > 1$, one $|\lambda| < 1$



d. $\lambda_1 = 0.5 + 0.6i, \lambda_2 = 0.5 - 0.6i.$

$\sqrt{.5^2 + .6^2} = \sqrt{.61} = .78 < 1$

attractor



6. For the vectors $\vec{u} = \begin{bmatrix} 1 \\ 9 \\ -3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$, find the following: (4 points each)

a. $||\vec{v}||$

$\sqrt{16 + 25 + 1} = \sqrt{42}$

b. A unit vector in the direction of $\vec{u}$.

$\sqrt{1 + 81 + 9} = \sqrt{91}$

$$\begin{bmatrix} 1/\sqrt{91} \\ 9/\sqrt{91} \\ -3/\sqrt{91} \end{bmatrix}$$

c. $\vec{u} \cdot \vec{v}$

$4 + 45 + 3 = 52$

d. Are the two vectors orthogonal? Why or why not?

no, the dot product is zero if the vectors are orthogonal

Part II:

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

7. Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$. (12 points)

$$(4-\lambda)\begin{vmatrix} -\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} \cancel{- 0\begin{vmatrix} -1 & 4 \\ 0 & 3-\lambda \end{vmatrix}} - 1\begin{vmatrix} -1 & -\lambda \\ 0 & 2 \end{vmatrix}$$

$$(4-\lambda)[-\lambda(3-\lambda)-8] - [-2-0] =$$

$$(4-\lambda)[-3\lambda+\lambda^2-8] + 2 = 4\lambda^2 - 12\lambda - 32 + 3\lambda^2 - \lambda^3 + 8\lambda + 2 =$$

$$-\lambda^3 + 7\lambda^2 - 4\lambda - 30 = 0 \quad \rightarrow \quad \lambda^3 - 7\lambda^2 + 4\lambda + 30 = 0$$

$\lambda = 5$

$$\begin{array}{r} \lambda^2 - 2\lambda - 6 \\ \lambda - 5 \,\big)\, \lambda^3 - 7\lambda^2 + 4\lambda + 30 \\ -\lambda^3 + 5\lambda^2 \qquad\qquad \\ \hline -2\lambda^2 + 4\lambda \qquad \\ +2\lambda^2 \mp 10\lambda \qquad \\ \hline -6\lambda + 30 \\ -6\lambda + 30 \\ \hline 0 \end{array}$$

$$(\lambda-5)(\lambda^2-2\lambda-6)$$

$$\lambda = \frac{2 \pm \sqrt{4+24}}{2} = 1 \pm \sqrt{7}$$

$$\lambda_1 = 5 \quad \begin{bmatrix} -1 & 0 & -1 \\ -1 & -5 & 4 \\ 0 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 \\ x_3 &= x_3 \end{aligned} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \pm \sqrt{7} \quad \begin{bmatrix} 3-\sqrt{7} & 0 & -1 \\ -1 & -1-\sqrt{7} & 4 \\ 0 & 2 & 2-\sqrt{7} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3/2-\sqrt{7}/2 \\ 0 & 1 & 1-\sqrt{7}/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 &= \frac{(-3-\sqrt{7})}{2} x_3 \\ x_2 &= \frac{(2-\sqrt{7})}{2} x_3 \\ x_3 &= x_3 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} -3-\sqrt{7} \\ 2-\sqrt{7} \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3+\sqrt{7} \\ 2+\sqrt{7} \\ 2 \end{bmatrix}$$

8. Find the similarity transformation for the matrix $B = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$ that converts this matrix into a similar rotation matrix. Then use that matrix to find the angle of rotation. Give your angle in radians rounded to 4 decimal places, or in degrees rounded to one decimal place. (12 points)

$$(4-\lambda)(6-\lambda)+2=0$$
$$24-4\lambda-6\lambda+\lambda^2+2=0$$
$$\lambda^2-10\lambda+26=0$$
$$\lambda = \frac{10 \pm \sqrt{100-104}}{2} = 5 \pm i$$
$$\lambda = 5-i$$
$$C = \begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix}$$
$$x_1 = -(1+i)x_2$$
$$x_2 = \quad x_2$$
$$\vec{v_2} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$$
$$P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$r = \sqrt{25+1} = \sqrt{26}$$
$$\cos^{-1}\left(5/\sqrt{26}\right) = 0.1974 \text{ radians or } 11.3°$$

9. Use your calculator to find the equilibrium vector of the stochastic matrix $P = \begin{bmatrix} .6 & .4 & .1 & .2 \\ .1 & .6 & .2 & .3 \\ .1 & 0 & .5 & .1 \\ .2 & 0 & .2 & .4 \end{bmatrix}$.

Explain the steps you took to obtain the vector. Then demonstrate that it is the correct equilibrium vector by multiplying by P. (10 points)

$$\vec{q} = \begin{bmatrix} 7/17 \\ 5/17 \\ 2/17 \\ 3/17 \end{bmatrix}$$

$$P\vec{q} = \begin{bmatrix} .6 & .4 & .1 & .2 \\ .1 & .6 & .2 & .3 \\ .1 & 0 & .5 & .1 \\ .2 & 0 & .2 & .4 \end{bmatrix} \begin{bmatrix} 7/17 \\ 5/17 \\ 2/17 \\ 3/17 \end{bmatrix} = \begin{bmatrix} 7/17 \\ 5/17 \\ 2/17 \\ 3/17 \end{bmatrix} = q$$

10. Does the matrix in #9 have enough communication between states to have only one equilibrium vector? Does the matrix have any absorbing states? Explain your reasoning. (5 points)

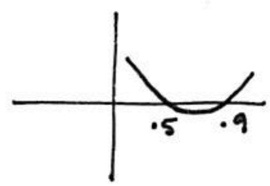
yes. there is only one equilibrium vector so this implies there is communication.

there are no absorbing states since none of the entries of P are 1.

11. Solve the discrete dynamical system given by $\vec{x}_{k+1} = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix} \vec{x}_k$. Find the eigenvalues and eigenvectors. (10 points)

$$(0.3 - \lambda)(1.1 - \lambda) + 0.12 = 0$$

$\lambda = .5 \quad \lambda = .9$ (obtained from graph)



$\lambda_1 = 0.5$

$$\begin{bmatrix} -.2 & .4 \\ -.3 & .6 \end{bmatrix}$$

$$-.2x_1 = -.4x_2$$

$$\begin{aligned} x_1 &= 2x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 0.9$

$$\begin{bmatrix} -.6 & .4 \\ -.3 & +.2 \end{bmatrix}$$

$$-.3x_1 = -.2x_2$$

$$\begin{aligned} x_1 &= \tfrac{2}{3}x_2 \\ x_2 &= x_2 \end{aligned} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{X}_{k+1} = c_1\lambda_1^{k+1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\lambda_2^{k+1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= c_1 0.5^{k+1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 0.9^{k+1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

c_1, c_2 determined by initial condition

12. Solve the system of ODEs given by $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\vec{x}$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point? Give your final solution in exact form with e. (15 points)

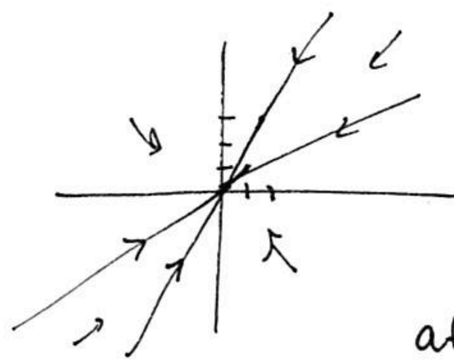
$(1-\lambda)(-4-\lambda)+6=0$

$-4-\lambda+4\lambda+\lambda^2+6=0$

$\lambda^2+3\lambda+2=0$

$(\lambda+2)(\lambda+1)=0$

$\lambda=-2,-1$



attractor

$\lambda=-2, \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ $\quad \lambda=-1 \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$

$3x_1 = 2x_2$ $\quad 3x_1 = 3x_2$

$x_1 = \frac{2}{3}x_2$ $\quad x_1 = x_2$

$x_2 = x_2$ $\quad x_2 = x_2$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 e^{-2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

13. The vectors $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal to each other. Find a vector $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ orthogonal to both. (10 points)

$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \to \text{rref} \to \begin{bmatrix} 1 & 0 & 2/13 & 3/13 \\ 0 & 1 & -3/13 & 2/13 \end{bmatrix}$$

$x_1 = -2/13\, x_3 - 3/13\, x_4$

$x_2 = 3/13\, x_3 - 2/13\, x_4$

$x_3 = x_3$

$x_4 = x_4$

$$\vec{w} = \begin{bmatrix} -2 \\ 3 \\ 13 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ -2 \\ 0 \\ 13 \end{bmatrix} s$$

e.g. either of these vectors or if $t=1, s=1$

$$\vec{w} = \begin{bmatrix} -5 \\ 1 \\ 13 \\ 13 \end{bmatrix}$$

14. Determine if the polynomials $p(t) = 2 - t, q(t) = t^2 + 2t$ are orthogonal under the inner product $\langle f|g\rangle = \int_{-2}^{2} f(t)g(t)dt$. (6 points)

$$\int_{-2}^{2} (2-t)(t^2+2t)\,dt = \int_{-2}^{2} \cancel{2t^2} + 4t - t^3 - \cancel{2t^2}\,dt =$$

$$\int_{-2}^{2} 4t - t^3\,dt = 0$$

odd

yes, they are orthogonal

15. Use least squares to find a linear regression equation $\beta_0 + \beta_1 x = y$ for the data shown in the table below. Be sure to write the final regression equation. Round your answers to two decimal places. (10 points)

x	11	12	13	15	18	18
y	42	43	44	52	84	87

$$A = \begin{bmatrix} 1 & 11 \\ 1 & 12 \\ 1 & 13 \\ 1 & 15 \\ 1 & 18 \\ 1 & 18 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 42 \\ 43 \\ 44 \\ 52 \\ 84 \\ 87 \end{bmatrix} \qquad (A^TA)^{-1}A^T\vec{b} = \hat{x} = \begin{bmatrix} -38.212 \\ 6.681 \end{bmatrix}$$

$$y = -38.21 + 6.68x$$

16. Let $W = span\left\{\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}, \begin{bmatrix}-4\\1\\0\\3\end{bmatrix}\right\}$. Given $\vec{y} = \begin{bmatrix}2\\-1\\3\\8\end{bmatrix}$, decompose this vector into $\vec{y}_\parallel$ in W and $\vec{y}_\perp$ in $W^\perp$. (10 points)

$u_1 \quad u_2$

$$\frac{u_1\cdot y}{u_1\cdot u_1}u_1 + \frac{u_2\cdot y}{u_2\cdot u_2}u_2 = \frac{2+2-3+16}{1+4+1+4}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{-8-1+0+24}{16+1+0+9}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \frac{17}{10}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{15}{26}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix}$$

$$y_{\parallel} = \begin{bmatrix}-79/130\\-367/130\\-17/10\\667/130\end{bmatrix} \approx \begin{bmatrix}-.60769\\-2.82308\\-1.7\\5.13078\end{bmatrix}$$

$$y_\perp = y - y_\parallel = \begin{bmatrix}2\\-1\\3\\8\end{bmatrix} - \begin{bmatrix}-79/130\\-367/130\\-17/10\\667/130\end{bmatrix} = \begin{bmatrix}339/130\\237/130\\47/10\\373/130\end{bmatrix} \approx \begin{bmatrix}2.60769\\1.82308\\4.7\\2.86923\end{bmatrix}$$

17. The vectors $\vec{u} = \begin{bmatrix}2\\-3\\1\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}3\\2\\0\\1\end{bmatrix}$ are orthogonal to each other. Let $W = span\{\vec{u}, \vec{v}\}$. Find an orthogonal basis for $W^\perp$. (10 points)

$$\begin{bmatrix}2 & -3 & 1 & 0\\3 & 2 & 0 & 1\end{bmatrix} \to \text{rref} \to \begin{bmatrix}1 & 0 & 2/13 & 3/13\\0 & 1 & -3/13 & 2/13\end{bmatrix}$$

$$\begin{aligned} x_1 &= -2/13\,x_3 - 3/13\,x_4 \\ x_2 &= 3/13\,x_3 - 2/13\,x_4 \\ x_3 &= x_3 \\ x_4 &= \qquad\qquad x_4 \end{aligned} \qquad \vec{x} = \begin{bmatrix}-2\\3\\13\\0\end{bmatrix}t + \begin{bmatrix}-3\\-2\\0\\13\end{bmatrix}s$$

$$w = \begin{bmatrix}-2\\3\\13\\0\end{bmatrix} \qquad \begin{bmatrix}2 & -3 & 1 & 0\\3 & 2 & 0 & 1\\-2 & 3 & 13 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 3/13\\0 & 1 & 0 & 2/13\\0 & 0 & 1 & 0\end{bmatrix}$$

$$y = \begin{bmatrix}-3\\-2\\0\\13\end{bmatrix}$$

$$W_\perp = span\left\{\begin{bmatrix}-2\\3\\13\\0\end{bmatrix}, \begin{bmatrix}-3\\-2\\0\\13\end{bmatrix}\right\}$$

Part III:

The following problems are "comprehensive" in that they cover the material from previous exams. You do not need to complete all the problems, but you should aim to earn a minimum of 40-50 points from this section.

18. Determine if the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -3x_2 \\ 2x_1 - 5x_2 \end{bmatrix}$ is linear or not. If it is, prove it. If it is not, find a counterexample. (6 points)

yes. $T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ linear

$$Tx + Ty = \begin{bmatrix} x_1 - 2x_2 \\ -3x_2 \\ 2x_1 - 5x_2 \end{bmatrix} + \begin{bmatrix} y_1 - 2y_2 \\ -3y_2 \\ -2y_1 - 5y_2 \end{bmatrix} = \begin{bmatrix} (x_1+y_1) - 2(x_2+y_2) \\ -3(x_2+y_2) \\ 2(x_1+y_1) - 5(x_2+y_2) \end{bmatrix} = T(x+y) \text{ closed under addition}$$

$$KT\cancel{x} = k\begin{bmatrix} x_1 - 2x_2 \\ -3x_2 \\ 2x_1 - 5x_2 \end{bmatrix} = \begin{bmatrix} k_1 - 2kx_2 \\ -3kx_2 \\ 2kx_1 - 5kx_2 \end{bmatrix} = T(k\vec{x}) \text{ closed under multiplication}$$

19. Given that A and B are $n \times n$ matrices with det A = -2 and det B = 7, find the following. (4 points each)

a) det AB -14

b) det A^{-1} $-\frac{1}{2}$

d) det $(-AB^3)$ $(-1)^n(-2)7^3 = (-1)^{n+1}686$

e) det 5A $5^n(-2)$

20. Suppose matrix A is a 5×9 matrix with 5 pivot columns. Determine the following. (12 points)

dim Col A = 5

dim Nul A = 4

dim Row A = 5

If Col A is a subspace of $\mathbb{R}^m$, then m = 5

Rank A = 5

If Nul A is a subspace of $\mathbb{R}^n$, then n = 9

21. Determine if the following sets are subspaces. Be sure to check all the necessary conditions or find a counterexample. (6 points each)

a. The set of complex numbers, of the form $z = a + bi$.

yes

$a, b = 0 \Rightarrow 0$ in set

$(a+bi) + (c+di) = (a+c) + (b+d)i$ closed

$k(a+bi) = (ka) + (kb)i$ closed

is a subspace

b. Polynomials of the form $p(t) = a + bt + ct^3$ as a subspace of P_3.

$a, b, c = 0 \Rightarrow p(t) = 0$ in set

$p(t) + q(t) = (a + bt + ct^3) + (d + et + ft^3) =$

$(a+d) + (b+e)t + (c+f)t^3$ in set, closed

$kp(t) = k(a + bt + ct^3) = (ka) + (kb)t + (kc)t^3$ closed

is a subspace

22. If a basis for R^3 is $B = \left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}$, and if a vector in the standard basis is $\vec{x} = \begin{bmatrix}2\\-4\\5\end{bmatrix}$, find its representation in the basis. (10 points)

$P_B [x]_B = \vec{x}$

$P_B^{-1} \vec{x} = [x]_B$

$$P_B = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -3 \\ 3 & -2 & 0 \end{bmatrix}$$

$$P_B^{-1} = \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 9/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix}$$

$$P_B^{-1} \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$