Math 266, Exam #2, Fall 2021 Name KEY

Part I:

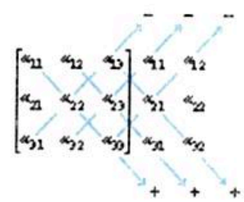
Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. You must show all hand written work on this part of the exam. Answers with no work will receive only 1 point. When you are finished with this portion of exam, continue with Part II.

1. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (3 points each)

a. T F If A is onto, then A is one-to-one.

b. T F If there is a matrix C so that $CA = I$, then $Rank\ A = n$.

c. T F The method of finding the determinant of a 3x3 matrix shown in the attached image generalizes to any size matrix.



d. T F Row reducing a matrix does not change its determinant.

e. T F A system that does not have a unique solution cannot be solved with Cramer's rule.

f. T F A matrix is invertible if the determinant of the matrix is 0.

g. T F Matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

h. T F The function $f(x) = 0$ is a subspace of P_n.

i. T F A vector is any element of a vector space. Specifically, a polynomial is a vector because the set of all polynomials with highest degree n, P_n, is a vector space.

j. T F R^3 is a subspace of R^4.

k. T F If two spaces have the same number of basis vectors, then then are isomorphic.

l. T F The column space of an $m \times n$ matrix is a subspace of R^m.

m. T F A vector space has infinite dimensions if there is no finite basis for the space.

2. Find the determinant of the matrix $\begin{bmatrix} 6 & 3 & 2 & 2 & 0 \\ 5 & 0 & -4 & 1 & 0 \\ 4 & -5 & 3 & 7 & 2 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$ by the cofactor method. (25 points)

$$2\begin{vmatrix} 6 & 3 & 2 & 2 \\ 5 & 0 & -4 & 1 \\ 3 & 0 & 0 & 1 \\ 4 & 2 & 3 & 2 \end{vmatrix} = 2\begin{bmatrix} 3 & 2 & 2 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{bmatrix}(3) - 2\begin{bmatrix} 6 & 3 & 2 \\ 5 & 0 & -4 \\ 4 & 2 & 3 \end{bmatrix}(1)$$

$$= 6\begin{vmatrix} 3 & 2 & 2 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 6 & 3 & 2 \\ 5 & 0 & -4 \\ 4 & 2 & 3 \end{vmatrix} =$$

$$6\left[3\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ -4 & 1 \end{vmatrix}\right] - 2\left[-3\begin{vmatrix} 5 & -4 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 6 & 2 \\ 5 & -4 \end{vmatrix}\right] =$$

$$18(-8-3) + 12(2+8) + 6(15+16) + 4(-24-10) =$$

$$18(-11) + 12(10) + 6(31) + 4(-34) = -28$$

$$-198 + 120 + 186 - 136$$

3. Find the determinant of the matrix $\begin{bmatrix} 1 & -2 & 4 \\ 3 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ by the row-reducing method. (15 points)

$-3R_1 + R_2 \to R_2$ no change

$-2R_1 + R_3 \to R_3$ no change

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 7 & -14 \\ 0 & 4 & -7 \end{bmatrix}$$

$-4/7\, R_2 + R_3 \to R_3$ no change

$$\begin{matrix} 0 & -4 & 8 \\ 0 & 4 & -7 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 7 & -14 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1 \times 7 \times 1 = 7$$

4. Given that A and B are $n \times n$ matrices with det A = -3 and det B = 2, find the following. (5 points each)

a) det AB $(-3)(2) = -6$

d) det B^T 2

b) det A^{-1} $-\frac{1}{3}$

e) det 2A $2^n(-3)$

c) det $(-AB^4)$

$(-1)^n(3)(2)^4 = (-1)^n 48$

5. Determine if the following sets are subspaces. Be sure to check all the necessary conditions or find a counterexample. (8 points each)

a. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \leq 0 \right\}$. not a subspace

$u = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ $(-4)(5) \leq 0$ $v = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $(5)(-1) \leq 0$

$u + v = \begin{bmatrix} -4 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ but $(1)(4) > 0$ not in set

not closed under addition

b. The set of all odd functions, i.e. $f(-x) = -f(x)$.

$f(x) = 0$ $-(0) = 0$ Zero is in set

$f(x)$ is odd, $g(x)$ is odd

$f(x) + g(x) \rightarrow -f(x) + [-g(x)] = f(-x) + g(-x) \rightarrow$

$(f+g)x$ is odd $-[f(x) + g(x)] = (f+g)(-x)$

closed under addition

$-kf(x) = kf(-x)$ closed under multiplication

is a subspace

c. Polynomials of the form $p(t) = (t-2)(a + bt + ct^2)$ as a subspace of P_3.

is a subspace

$(t-2)(0 + 0t + 0t^2) = 0$ 0 is in set

$p(t) = (t-2)(a + bt + ct^2)$ $q(t) = (t-2)(d + et + ft^2)$

$p + q = (t-2)[(a+d) + (b+e)t + (c+f)t^2)$ closed under addition

$kp(t) = k(t-2)(a + bt + ct^2) = (t-2)(ka + kbt + kct^2)$ closed under multiplication

6. Suppose matrix A is a 6×8 matrix with 5 pivot columns. Determine the following. (18 points)

dim Col A = 5 dim Nul A = 3

dim Row A = 5 If Col A is a subspace of R^m, then m = 6

Rank A = 5 If Nul A is a subspace of R^n, then n = 8

Part II:

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

7. If a basis for R^3 is $B = \left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}$, and given $[\vec{x}]_B = \begin{bmatrix}1\\0\\-2\end{bmatrix}$, find $\vec{x}$ in the standard basis. (6 points)

$P_B [x]_B = \vec{x}$

$$\begin{bmatrix}1 & 3 & 4\\-1 & 0 & -3\\3 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix} = \begin{bmatrix}-7\\5\\3\end{bmatrix}$$

8. If a vector in the standard basis is $\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$, find its representation in the basis in problem #7. (7 points)

$$P_B^{-1} \vec{x} = [x]_B$$

$$P_B^{-1} = \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 9/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix}$$

$$[x]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

9. Consider the basis $C = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$, and the vector $[\vec{x}]_C = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$. Find the representation of the vector in the basis B in problem #7. (8 points)

$$P_B [x]_B = P_C [x]_C$$

$$[x]_B = P_B^{-1} P_C [x]_C$$

$$= \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 9/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ -1 & 2 & -1 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7.96 \\ 4.44 \\ -4.32 \end{bmatrix} = \begin{bmatrix} 199/25 \\ 111/25 \\ -108/25 \end{bmatrix}$$

10. Use Cramer's rule to find the solution to the system $\begin{cases} x_1 + 3x_2 - 2x_3 + x_4 = 12 \\ 2x_1 - x_2 + x_3 + 4x_4 = 2 \\ 2x_2 - 3x_3 + 2x_4 = 12 \\ 3x_1 + 2x_3 - 5x_4 = -6 \end{cases}$. Write all the required matrices and their determinants, but you may calculate the determinants with your calculator. (15 points)

$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 0 & 2 & -3 & 2 \\ 3 & 0 & 2 & -5 \end{bmatrix}$ $\det A = -88$ $A_1 = \begin{bmatrix} 12 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 12 & 2 & -3 & 2 \\ -6 & 0 & 2 & -5 \end{bmatrix}$ $\det A_1 = -88$

$A_2 = \begin{bmatrix} 1 & 12 & -2 & 1 \\ 2 & 2 & 1 & 4 \\ 0 & 12 & -3 & 2 \\ 3 & -6 & 2 & -5 \end{bmatrix}$ $\det A_2 = -176$ $A_3 = \begin{bmatrix} 1 & 3 & 12 & 1 \\ 2 & -1 & 2 & 4 \\ 0 & 2 & 12 & 2 \\ 3 & 0 & -6 & -5 \end{bmatrix}$ $\det A_3 = 176$

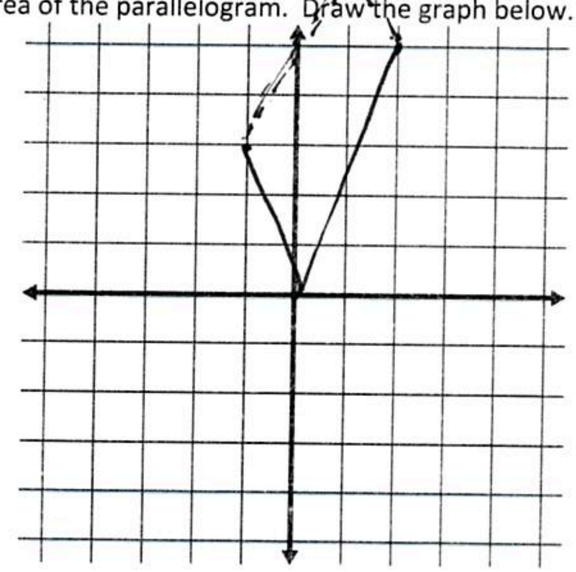
$A_4 = \begin{bmatrix} 1 & 3 & -2 & 12 \\ 2 & -1 & 1 & 2 \\ 0 & 2 & -3 & 12 \\ 3 & 0 & 2 & -6 \end{bmatrix}$ $\det A_4 = -88$

$x_1 = \frac{\det A_1}{\det A} = \frac{-88}{-88} = 1$ $x_4 = \frac{\det A_4}{\det A} =$

$x_2 = \frac{\det A_2}{\det A} = \frac{-176}{-88} = 2$ $\frac{-88}{-88} = 1$

$x_3 = \frac{\det A_3}{\det A} = \frac{176}{-88} = -2$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$

11. Suppose that a parallelogram is bounded at one vertex by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the area of the parallelogram. Draw the graph below. (6 points)



$\det\left(\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}\right) =$

$6 + 5 = 11$

12. A parallelepiped (slanted box) is defined in one corner by the vectors $\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \vec{v} = \begin{bmatrix}-1\\1\\1\end{bmatrix}, \vec{w} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$.

(8 points)

$$\det\left(\begin{bmatrix}1 & -1 & 3\\ 2 & 1 & -2\\ 3 & 1 & 5\end{bmatrix}\right) = 20$$

13. Find the nullspace of the system $\begin{cases} x_1 + 2x_2 - x_3 - 4x_4 + x_5 + 2x_6 = 0 \\ 4x_1 - 2x_2 + 3x_5 - x_6 = 0 \\ 2x_1 - x_3 + 2x_4 + 5x_6 = 0 \end{cases}$. (12 points)

$$\begin{bmatrix}1 & 2 & -1 & -4 & 1 & 2\\ 4 & -2 & 0 & 0 & 3 & -1\\ 2 & 0 & -1 & 2 & 0 & 5\end{bmatrix} \to \text{rref} \to \begin{bmatrix}1 & 0 & 0 & -2 & 4/3 & -4/3\\ 0 & 1 & 0 & -4 & 7/6 & -13/6\\ 0 & 0 & 1 & -6 & 8/3 & -23/3\end{bmatrix}$$

$x_1 = 2x_4 - 4/3\, x_5 + 4/3\, x_6$

$x_2 = 4x_4 - 7/6\, x_5 + 13/6\, x_6$

$x_3 = 6x_4 - 8/3\, x_5 + 23/3\, x_6$

$x_4 = x_4$

$x_5 = x_5$

$x_6 = x_6$

$$\vec{x} = \begin{bmatrix}2\\4\\6\\1\\0\\0\end{bmatrix}x_4 + \begin{bmatrix}-4/3\\-7/6\\-8/3\\0\\1\\0\end{bmatrix}x_5 + \begin{bmatrix}4/3\\13/6\\23/3\\0\\0\\1\end{bmatrix}x_6 = \begin{bmatrix}2\\4\\6\\1\\0\\0\end{bmatrix}t + \begin{bmatrix}-8\\-7\\-16\\0\\6\\0\end{bmatrix}s + \begin{bmatrix}8\\13\\46\\0\\0\\6\end{bmatrix}$$

14. Determine if the following sets of vectors are linearly independent. Then determine if they form a basis for the specified space. Explain your reasoning. (6 points each)

a. $\left\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}5\\0\\1\\2\end{bmatrix}, \begin{bmatrix}4\\-1\\0\\1\end{bmatrix}\right\}, R^4$

independent
spans space row equivalent to identity
is a basis for $\mathbb{R}^4$

b. $\{1 - t^2, 2 - 3t, 4t + t^2\}, P_2$

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}\begin{bmatrix}2\\-3\\0\end{bmatrix}\begin{bmatrix}0\\4\\1\end{bmatrix}$$

independent
spans space.
row equivalent to identity
is a basis for P_2