HW 4 P1

a) rref of augmented matrix:

$$\begin{bmatrix} 1 & 4 & 2 & \vdots & b_1 \\ 0 & 0 & 0 & \vdots & b_2 - 2b_1 \\ 0 & 0 & 0 & \vdots & b_3 + b_1 \end{bmatrix}$$

To have a solution $b_2 - 2b_1 = 0 \Rightarrow b_2 = 2b_1$

$b_3 - b_1 = 0 \Rightarrow b_3 = b_1$

Solutions exist iff $b_2 = 2b_1$ and $b_3 = b_1$

b) Not a square matrix $\Rightarrow$ no inverse

rref of augmented matrix: $\begin{bmatrix} 1 & 4 & \vdots & b_1 \\ 2 & 9 & \vdots & b_2 \\ 0 & 0 & \vdots & b_1 + b_3 \end{bmatrix}$

$\Rightarrow b_1 + b_3 = 0 \Rightarrow b_1 = -b_3$

Solutions iff $b_1 = -b_3$

HW 4 P2

2x2 matrix w/ nullspace = Col space

columns : $n=2$

rank : r

If nullspace = col space

$$n-r = r$$

$$n=2r$$

$$r=\frac{n}{2}=1$$

$\Rightarrow$ need rank-1 matrix such that $\underline{A}\,\underline{x}=\underline{0}$,

Let $\underline{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ w/ either $x_1 \neq 0$ and/or $x_2 \neq 0$

let $\underline{A}=[\underline{x}\ \ \underline{x}]$, then

$$\begin{bmatrix} x_1 & x_1 \\ x_2 & x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1^2+x_1x_2=0$$

$\Rightarrow x_1+x_2=0$ if $x_1 \neq 0$

$\Rightarrow x_2=-x_1$

Let $x_1=1$, then $x_2=-1$

$\Rightarrow \underline{A}=\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ note: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$, etc

are also correct.

HW 4 P3

Dimension Theorem for matrix $\underline{A} \in M_{nn}$

$$\text{Rank}(\underline{A}) + \text{nullity}(\underline{A}) = n$$

let $\text{Rank}(\underline{A}) = \text{Nullity}(\underline{A}) = r$

Then

$$r + r = n$$

$$2r = n$$

$$r = \frac{n}{2}$$

If $n = 3$ then $r = \frac{3}{2} \notin$ not possible

HW 4 p4

a) $\underline{A} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank $= 1$

nullity $= n - r = 4 - 1 = 3$

b) $\underline{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank $= 2$

nullity $= n - r = 4 - 2 = 2$

HW 4 p5

rank-1 means only 1 independent column

a) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$

$\underline{v} \quad 2\underline{v} \quad 4\underline{v}$

b) $\begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$

$\underline{v} \quad 3\underline{v} \quad -\frac{3}{2}\underline{v}$

c) $\begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}$

$\underline{v} \quad \frac{b}{a}\underline{v}$

HW4 P6

$$\underline{A} = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

If $c \neq 0$ then $\text{rank}(\underline{A}) = 3$, no matter the value of d, $\Rightarrow c = 0$

$$\Rightarrow \underline{A} = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

← need 1 pivot in this sub-matrix

True if $d = 2$

$$\Rightarrow \underline{A} = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ $\text{Rank}(\underline{A}) = 2$

$$\underline{B} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$ $$\text{rref}(\underline{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ $\text{Rank}(\underline{B}) = 2$

HW 4 p7

a) $\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

b) V is a subspace of $\mathbb{R}^3 \Rightarrow$ at most one vector which is orthogonal to V

$\underline{B} = [b_1 \ b_2 \ b_3]$

$\Rightarrow [b_1 \ b_2 \ b_3] \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = [0 \ 0]$

$\Rightarrow b_1 + b_2 + b_3 = 0$

$2b_1 + b_2 = 0 \quad \Rightarrow b_2 = -2b_1$

$\Rightarrow b_1 - 2b_1 + b_3 = 0$

$\Rightarrow -b_1 + b_3 = 0$

let $b_1 = 1$, then $b_3 = 1$ & $b_2 = -2$

$\underline{B} = [1 \ -2 \ 1]$ (and multiples of)

c) $\underline{A}\,\underline{B}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 1 \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

HW4 p8

a) $\underline{A} \in M_{mn}$, rank$(\underline{A}) = r \leq n$

If $\underline{A}\underline{x} = \underline{b}$ has no solution then $r < m$

Can't relate m & n

b) $\dim(N(\underline{A}^T)) = m - r > 0$ since $r < m$

Thus $N(\underline{A}^T)$ must contain at least one vector $\underline{y}$ such that $\underline{A}^T\underline{y} = \underline{0}$

HW4 P9

let $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\quad a_i, b_i \in \mathbb{R}^1$

$$\underline{a}\,\underline{b}^T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} (b_1 \quad b_2) = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$$

Divide row-1 by a_1 & row-2 by a_2 $\Rightarrow$

$$\begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix} \Rightarrow \quad \text{ref}(\underline{a}\,\underline{b}^T) = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{rank}(\underline{a}\,\underline{b}^T) = 1$$

let $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\underline{a}\,\underline{b}^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{array}{l} \leftarrow 1/a_1 \\ \leftarrow 1/a_2 \\ \leftarrow 1/a_3 \end{array}$$

$$\Rightarrow \begin{bmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{ref}(\underline{a}\,\underline{b}^T)$$

$$\Rightarrow \text{rank}(\underline{a}\,\underline{b}^T) = 1$$

Similar for $\mathbb{R}^n$, $n > 3$