HW 6 P2

$$\bar{\bar{A}} = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\bar{\bar{A}}^T\bar{\bar{A}} = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 9 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\det(\bar{\bar{I}}\lambda - \bar{\bar{A}}^T\bar{\bar{A}}) = \begin{vmatrix} \lambda-2 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & \lambda-9 & -2 \\ 0 & -2 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)[(\lambda-2)(\lambda-9)-4] - 8(\lambda-2)$$
$$= (\lambda-2)^2(\lambda-9) - 4(\lambda-2)$$
$$= (\lambda-2)[(\lambda-2)(\lambda-9)-4]$$
$$= (\lambda-2)[\lambda^2 - 11\lambda + 14]$$
$$\Rightarrow \lambda = 8, 2, 0 \quad \text{or} \quad \sigma = 2\sqrt{2}, \sqrt{2}, 0$$
$$\text{rank} = 2$$

$\lambda = 8$:

$$\begin{bmatrix} 2-8 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 9-8 & 2 \\ 0 & 2 & 2-8 \end{bmatrix} \sim \begin{bmatrix} -6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 2 \\ 0 & 2 & -6 \end{bmatrix}$$

$$\text{rref} \Rightarrow \begin{bmatrix} 1 & 0 & -\sqrt{2} \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \quad \bar{x}_1 = \begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \bar{v}_1 = \frac{\bar{x}_1}{\|\bar{x}_1\|} = \frac{1}{2\sqrt{5}}\bar{x}_1 = \begin{bmatrix} \sqrt{2}/\sqrt{5} \\ 3/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 0.4082 \\ 0.8660 \\ 0.2887 \end{bmatrix}$$

$\lambda = 2$:

$$\begin{bmatrix} 2-2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-2 & 2 \\ 0 & 2 & 2-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\text{rref} = \begin{bmatrix} 1 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{v}_2 = \frac{\underline{x}_2}{\|\underline{x}_2\|} = \frac{1}{\sqrt{3/2}} \underline{x}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ \sqrt{2/3} \end{bmatrix} \approx \begin{bmatrix} -0.5774 \\ 0 \\ 0.8165 \end{bmatrix}$$

$\lambda = 0$:

$$\begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\text{rref} = \begin{bmatrix} 1 & 0 & -\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{x}_3 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}, \underline{v}_3 = \frac{\underline{x}_3}{\|\underline{x}_3\|} = \frac{1}{2} \underline{x}_3 = \begin{bmatrix} \sqrt{2}/2 \\ -1/2 \\ 1/2 \end{bmatrix} \approx \begin{bmatrix} 0.7071 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$\Rightarrow \underline{\Sigma} = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{V} = \begin{bmatrix} 0.4082 & -0.5774 & 0.7071 \\ 0.8660 & 0 & -0.5 \\ 0.2887 & 0.8165 & 0.5 \end{bmatrix}$$

Now solve $\underline{U}\,\underline{\Sigma} = \underline{A}\,\underline{V}$

$$[\underline{u}_1 \;\; \underline{u}_2 \;\; \underline{u}_3] \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = [\sigma_1 \underline{u}_1 \;\; \sigma_2 \underline{u}_2 \;\; \sigma_3 \underline{u}_3]$$

$$\underline{A}\,\underline{V} = \begin{bmatrix} 1.1547 & 0.8165 & 0 \\ 2.3094 & -0.8165 & 0 \\ 1.1547 & 0.8165 & 0 \end{bmatrix} = [\sigma_1 \underline{u}_1 \;\; \sigma_2 \underline{u}_2 \;\; \sigma_3 \underline{u}_3]$$

$\underline{A}\underline{v}_1 \qquad \underline{A}\underline{v}_2$

$$\Rightarrow \underline{u}_1 = \frac{1}{\sigma_1}\underline{A}\underline{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix}$$

$$\underline{u}_2 = \frac{1}{\sigma_2}\underline{A}\underline{v}_2 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

$\underline{u}_3$ is in null space of $\underline{A}^T$

$$\text{rref}(\underline{A}^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x}_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{u}_3 = \frac{\underline{x}_4}{\|\underline{x}_4\|} = \frac{1}{\sqrt{2}}\underline{x}_4 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$$

$U \Sigma V^T$

$\Rightarrow \underline{A} =$

$$\underbrace{\begin{bmatrix} 0,4082 & 0,5774 & -0,7071 \\ 0,8165 & -0,5774 & 0 \\ 0,4082 & 0,5774 & 0,7071 \end{bmatrix}}_{\underline{U}} \underbrace{\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\Sigma}} *$$

$$\underbrace{\begin{bmatrix} 0,4082 & 0,8660 & 0,2887 \\ -0,5774 & 0 & 0,8165 \\ 0,7071 & -0,5 & 0,5 \end{bmatrix}}_{\underline{V}^T}$$

$$\text{Rank}(\underline{A}) = 2$$

Note: Using $\underline{A}\underline{A}^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

gives the same result.