MTH 267 Practice Exam #2 Key

1a. $2y'' - y' - y = 0$

$2r^2 - r - 1 = 0$

$(2r+1)(r-1) = 0 \rightarrow r = -\frac{1}{2}, 1$

$y(t) = c_1 e^{-1/2 t} + c_2 e^t$

b. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0 \rightarrow r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$y(t) = c_1 e^t \sin t + c_2 e^t \cos t$

c. $y'' - 18y' + 81y = 0$

$r^2 - 18r + 81 = 0$

$(r-9)^2 = 0 \rightarrow r = 9$ repeated

$y(t) = c_1 e^{9t} + c_2 t e^{9t}$

Ansatz = $Y(x)$ or Y_p

2a. $Y(x) = A \sin 3x + B \cos 3x$

b. $Y(x) = Ae^x \sin x + Be^x \cos x$

c. $Y(x) = Ae^x + B$

d. $Y(t) = Ate^{-t} + t(Ct + D) = Ate^{-t} + Ct^2 + Dt$

e. $Y(t) = A + B\cos 2t + C \sin 2t$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$

3. the natural frequency of the system is determined w/o damping. The quasi-frequency is the frequency of the decaying oscillation w/ damping included.

4. Beats occur when forcing and natural frequency functions have similar but not identical frequencies that sometimes act constructively and sometimes destructively as they fall in and out of sync.

5. $(1-x^2)y'' - 2xy' + 2y = 0 \qquad y_1(x) = x$

$y_2(x) = v(x) \cdot y_1 = vx$

$y_2' = v + xv'$

$y_2'' = 2v' + xv''$

$(1-x^2)(2v' + xv'') - 2x(v + xv') + 2xv = 0$

$2v' + xv'' - 2x^2v' - x^3v'' - \cancel{2xv} - 2x^2v' + \cancel{2xv} = 0$

$2v' + xv'' - 4x^2v' - x^3v'' = 0$

5 cont'd.

$v''(x - x^3) + v'(2 - 4x^2) = 0$ $\qquad u = v' \quad \frac{du}{dx} = v''$

$v''(x - x^3) = (4x^2 - 2)v'$

$\frac{du}{u} = \frac{4x^2 - 2}{x - x^3}$ $\qquad \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$

$x(1 - x^2) = x(1-x)(1+x)$ $\qquad A - Ax^2 + Bx + Bx^2 + Cx - Cx^2 = 4x^2 - 2$

$\int \frac{du}{u} = \int \frac{-2}{x} + \frac{1}{1-x} - \frac{1}{1+x}\,dx$ $\qquad -A + B - C = 4$

$B + C = 0 \qquad A = -2$

$\ln u = -2\ln x - \ln|1-x| - \ln|1+x|$ $\qquad 2 + B - C = 4$

$= \ln\left|\frac{1}{x^2}\cdot\frac{1}{1-x}\cdot\frac{1}{1+x}\right|$

$B - C = 2$

$B + C = 0$

$2B = 2 \Rightarrow B = 1 \Rightarrow C = -1$

$\Rightarrow u = \frac{1}{x^2(1-x)(1+x)} = v'$ $\qquad \Rightarrow v = \int \frac{1}{x^2(1-x)(1+x)}\,dx$

$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x} = \frac{1}{x^2(1-x)(1+x)}$

$Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x) =$

$Ax - Ax^3 + B - Bx^2 + Cx^2 + Cx^3 + Dx^2 - Dx^3 = 1$

$-A + C - D = 0 \qquad C - D = 0$ $\qquad v = \int \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{x+1}\,dx$

$-B + C + D = 0 \qquad C + D = 1$

$2C = 1$

$-A = 0$ $\qquad v =$

$-B + 1 = 0 \Rightarrow B = 1 \qquad C = 1/2$ $\qquad \Rightarrow -\frac{1}{x} - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|x+1| =$

$D = 1/2$

$-\frac{1}{x} + \ln\sqrt{\frac{x+1}{1-x}} = v$

$y_2 = x\left(-\frac{1}{x} + \ln\sqrt{\frac{x+1}{1-x}}\right) = -1 + x\ln\sqrt{\frac{x+1}{1-x}}$

general solution: $y = c_1 x + c_2\left(-1 + x\ln\sqrt{\frac{x+1}{1-x}}\right)$

6. $12 = k(1/2) \leftarrow 6 \text{ in} = 1/2 \text{ ft}$

$k = 24 \qquad my'' + \gamma y' + ky = F(t) \qquad y(0) = -1,\ y'(0) = 0$

$\gamma = 3 \qquad \frac{3}{8}y'' + 3y' + 24y = 0 \qquad r = \frac{-8 \pm \sqrt{64 - 4(64)}}{2} =$

$12 = 32m \qquad 3y'' + 24y' + 192y = 0$

$m = \frac{3}{8}$ slugs $\qquad y'' + 8y' + 64 = 0 \qquad \frac{-8 \pm 8\sqrt{3}i}{2} = -4 \pm 4\sqrt{3}i$

$r^2 + 8r + 64 = 0$

6 cont'd. (3)

a. underdamped

b. $y(t) = c_1 e^{-4t}\sin(4\sqrt{3}t) + c_2 e^{-4t}\cos(4\sqrt{3}t)$

$y(0) = 1 = \cancel{c_1(1)(0)}^{\,0} + c_2(1)(1) \Rightarrow c_2 = 1$

$y'(t) = -4c_1e^{-4t}\sin(4\sqrt{3}t) + 4\sqrt{3}c_1e^{-4t}\cos(4\sqrt{3}t) - 4e^{-4t}\cos(4\sqrt{3}$

$-4\sqrt{3}e^{-4t}\sin(4\sqrt{3}t)$

$y'(0) = 0 = \cancel{4c_1(1)(0)}^{\,0} + 4\sqrt{3}c_1(1)(1) - 4(1)(1) \cancel{-4\sqrt{3}(1)(0)}^{\,0}$

$\Rightarrow 4\sqrt{3}c_1 - 4 = 0 \Rightarrow 4\sqrt{3}c_1 = 4 \Rightarrow c_1 = \frac{1}{\sqrt{3}}$

$y(t) = \frac{1}{\sqrt{3}}e^{-4t}\sin(4\sqrt{3}t) + e^{-4t}\cos(4\sqrt{3}t)$

c. $T = \frac{2\pi}{\omega} = \frac{2\pi}{4\sqrt{3}}$ quasi-period

amplitude (at start) $\sqrt{\frac{1}{\sqrt{3}}^2 + 1^2} = \sqrt{\frac{1}{3}+1} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{1/\sqrt{3}}\right) = \tan^{-1}(\sqrt{3}) = \pi/3$ phase shift

d. as $t \to \infty$, $y \to 0$ oscillation decays exponentially

7. $y'' + 6y' + 9y = 4e^{2t} + e^{-t}$ $\quad y_1 = e^{-3t}$, $y_2 = te^{-3t}$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0 \Rightarrow r = -3$

$$W = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t} - \cancel{3te^{-6t}} + \cancel{3te^{-6t}} = e^{-6t}$$

$$Y(t) = -e^{-3t}\int \frac{(4e^{2t}+e^{-t})te^{-3t}}{e^{-6t}}\,dt + te^{-3t}\int \frac{(4e^{2t}+e^{-t})e^{-3t}}{e^{-6t}}\,dt$$

$= -e^{-3t}\int e^{6t}(4te^{-t} + te^{-4t})\,dt + te^{-3t}\int e^{6t}(4e^{-t} + e^{-4t})\,dt$

$= e^{-3t}\int 4te^{5t} + te^{2t}\,dt + te^{-3t}\int 4e^{5t} + e^{2t}\,dt =$

$e^{-3t}\left[\frac{4}{5}te^{5t} - \frac{4}{25}e^{5t} + \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t}\right] + te^{-3t}\left[\frac{4}{5}e^{5t} + \frac{1}{2}e^{2t}\right]$

$= -\cancel{\frac{4}{5}te^{2t}} + \frac{4}{25}e^{2t} - \cancel{\frac{t}{2}e^{-t}} + \frac{1}{4}e^{-t} + \cancel{\frac{4}{5}te^{2t}} + \cancel{\frac{1}{2}te^{-t}} = \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$

$Y_p = c_1e^{-3t} + c_2te^{-3t} + \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$

8. $Y_1 = e^{-3t}$, $Y_2\ te^{-3t}$ (See above)

$Y(t) = Ae^{2t} + Be^{-t}$ $\quad Y'(t) = 2Ae^{2t} - Be^{-t}$ $\quad Y''(t) = 4Ae^{2t} + Be^{-t}$

$4Ae^{2t} + Be^{-t} + 6(2Ae^{2t} - Be^{-t}) + 9(Ae^{2t} + Be^{-t}) = 4e^{2t} + e^{-t}$

$4Ae^{2t} + Be^{-t} + 12Ae^{2t} - 6Be^{-t} + 9Ae^{2t} + 9Be^{-t} = 4e^{2t} + e^{-t}$

$4A + 12A + 9A = 25A = 4 \Rightarrow A = \frac{4}{25}$

$B - 6B + 9B = 4B = 1 \Rightarrow B = \frac{1}{4}$

$Y(t) = \frac{4}{25}e^{2t} + \frac{1}{4}e^{-t}$ (agrees w/ #7)

9. $Y(t) = At^2 + Bt + C$ $\quad Y'(t) = 2At + B$ $\quad Y''(t) = 2A$

$2(2A) + 3(2At + B) + At^2 + Bt + C = t^2$

$A = 1$ $\quad 6A + B = 0 \Rightarrow B = -6$ $\quad 4A + 3B + C = 0 \Rightarrow 4 - 18 + C = 0 \Rightarrow C = 14$

$Y_1(t) = t^2 - 6t + 14$

$Y_2(t) = D\sin t + E\cos t$ $\quad Y_2'(t) = D\cos t - E\sin t$ $\quad Y_2''(t) = -D\sin t - E\cos t$

$2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0$ $\quad r = -\frac{1}{2}, -1$ $\quad y_g(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t}$

$2(-D\sin t - E\cos t) + 3(D\cos t - E\sin t) + D\sin t + E\cos t = 3\sin t$

$(-2D - 3E + D)\sin t + (-2E + 3D + E)\cos t = 3\sin t$

$-D - 3E = 3$

$-E + 3D = 0 \Rightarrow E = 3D$ $\quad\quad Y_2(t) = -\frac{3}{10}\sin t - \frac{9}{10}\cos t$

$-D - 3(3D) = 3$ $\quad E = -\frac{9}{10}$

$-10D = 3$

$D = -3/10$

$Y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t$

10. $y'' - 2y' + y = 0$ $\quad Y_1 = e^t$, $Y_2 = te^t$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$ $\quad W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = e^{2t}$

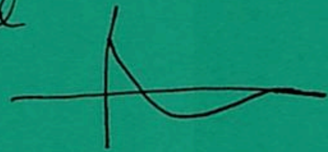
$r = 1$ repeated

$Y(t) = -e^t \int \frac{t}{1+t^2}\,dt + te^t \int \frac{1}{1+t^2}\,dt = -\frac{1}{2}e^t \ln|1+t^2| + te^t \arctan t$

$y_p(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln|1+t^2| + te^t \arctan t$

11. a. resonance $\Rightarrow$ $t \sin t$ term   b. beats. freq. are 6, 7   ⑤

12. overdamped



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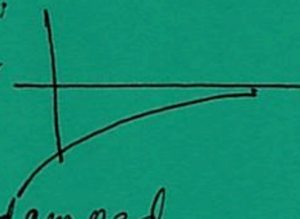

goes directly to 0
or crosses origin only once

13. a. transient: e^{-t} terms, underdamped
steady state: are $5\cos 4t + 4\sin 4t$
no resonance or beats

b. transient: first 2 terms e^{-rt} overdamped
steady state: $\sin 3t$ no resonance or beats

c. all steady state, undamped, resonance

14. $W = e^{-\int 2x\,dx} = e^{-x^2}$

15. $W = \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t\ln t + t \end{vmatrix} = \cancel{2t^3 \ln t} + t^3 - \cancel{2t^3 \ln t} = t^3$

16. $9.8(1/10) = k(0.05) \Rightarrow k = 19.6$

a. $y(0) = 0$
$y'(0) = -\frac{1}{10}$

$\frac{1}{10} y'' + 19.6 y = 0$
$y'' + 196 y = 0$
$r^2 + 196 = 0 \rightarrow r = \pm 14i$

$y(t) = c_1 \sin 14t + c_2 \cos 14t$ $y(0) = \cancel{c_1(0)}^{0} + c_2(1) = 0$
$c_2 = 0$

$y(t) = c_1 \sin 14t \rightarrow y'(t) = 14c_1 \cos 14t \Rightarrow y'(0) = -\frac{1}{10} = 14c_1(1)$
$\Rightarrow c_1 = -\frac{1}{140}$

$y(t) = -\frac{1}{140} \sin 14t$

b. $14t = \pi \Rightarrow t = \frac{\pi}{14} \approx .2244$ seconds

c. $P = \frac{2\pi}{14} = \frac{\pi}{7}$ period
amplitude $= \frac{1}{140}$
phase shift $= 0$

17. $\frac{dA}{dt} = \frac{1\text{ lbs}}{1\text{ gal}} \cdot \frac{5\cancel{\text{gal}}}{1\text{ sec}} - \frac{A\text{ lbs}}{100+2t\ \text{gal}} \cdot \frac{3\cancel{\text{gal}}}{1\text{ sec}}$ $A(0) = 50$

$\Rightarrow 100 + (\Delta\text{flow})t$ • if rate in & out of water is same this is 0t

17 cont'd.

$$\frac{dA}{dt} = 5 - \frac{3A}{100+2t} \Rightarrow \frac{dA}{dt} + \frac{3}{100+2t}A = 5$$

$$\mu = e^{\int \frac{3}{100+2t}dt} = e^{\frac{3}{2}\ln(100+2t)} = (100+2t)^{3/2}$$

$$(100+2t)^{3/2}\frac{dA}{dt} + 3(100+2t)^{1/2}A = 5(100+2t)^{3/2}$$

$$\int\left((100+2t)^{3/2}A\right)' = \int 5(100+2t)^{3/2}\,dt$$

$$(100+2t)^{3/2}A = \frac{\cancel{5}\cdot\frac{2}{\cancel{5}}}{\cancel{2}}(100+2t)^{5/2} + C$$

$$A = 100+2t + \frac{C}{(100+2t)^{3/2}} \Rightarrow 50 = 100+2(0) + \frac{C}{(100+2(0))^{3/2}}$$

$$\Rightarrow -50 = \frac{C}{(100)^{3/2}} \Rightarrow C = -50(1000) = -50{,}000$$

a. $A(t) = 100+2t - \dfrac{50{,}000}{(100+2t)^{3/2}}$ $100+2t = 400 \Rightarrow 2t = 300$
$\Rightarrow t = 150$

can fill for 150 seconds before the tank overflows

b. $A(150) = 100+2(150) - \dfrac{50{,}000}{(100+2(150))^{3/2}} = 393.75$ lbs of salt

c. concentration $= \dfrac{\text{total salt}}{\text{amount of water}}$

$$C(t) = \frac{A(t)}{100+2t}$$

d. when tank is full $\underset{C(150)}{C(t)} = 0.984375$ lbs/gal

e. 90% of $0.984375 = 0.8859375$

$$C(t) = \frac{100+2t - 50{,}000/(100+2t)^{3/2}}{100+2t} = 1 - \frac{50{,}000}{(100+2t)^{5/2}}$$

$$1 - \frac{50{,}000}{(100+2t)^{5/2}} = 0.8859375 \Rightarrow t = 40.3 \text{ seconds}$$

18. $k(T-72) = \frac{dT}{dt}$

$\int k\,dt = \int \frac{dT}{T-72}$

$kt + C = \ln|T-72|$

$A_0 e^{kt} = T - 72$

$T = 72 + A_0 e^{kt}$

$T(0) = 200$

$A_0 = 128$

$T(1) = 180$

$180 = 72 + 128 e^{k(1)}$

$\frac{108}{128} = e^k \Rightarrow k = -0.1699$

$T(t) = 72 + 128 e^{-0.1699t}$

$\Rightarrow 120 = 72 + 128 e^{-0.1699t}$

$\Rightarrow t = 5.77$ minutes

20. half-life = 11.2 hrs.

$\frac{dA}{dt} = kA$

$A_t = A_0 e^{kt}$ peak 100%

$50\% = 100\% e^{k(11.2)} \Rightarrow k = -0.061888$

$5\% = 100\% e^{-0.061888t}$

$\Rightarrow t = 48.4$ hrs