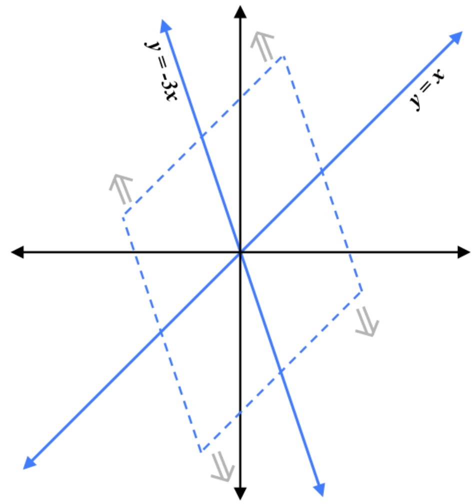
The Stretching Problem

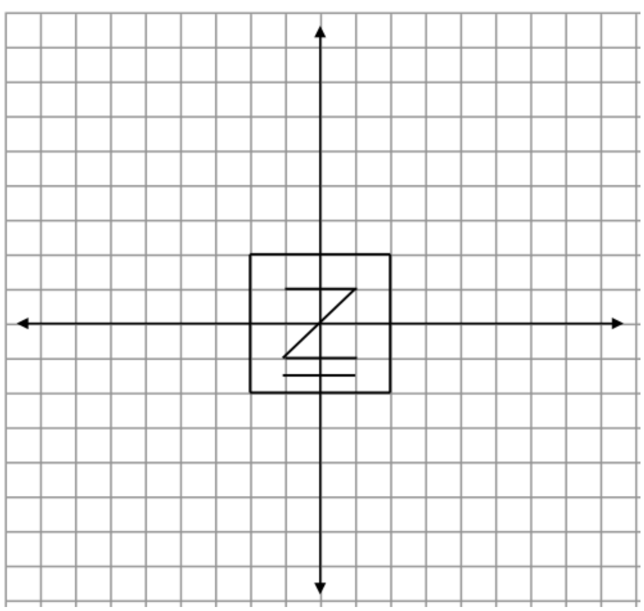
Imagine a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that has the following properties:

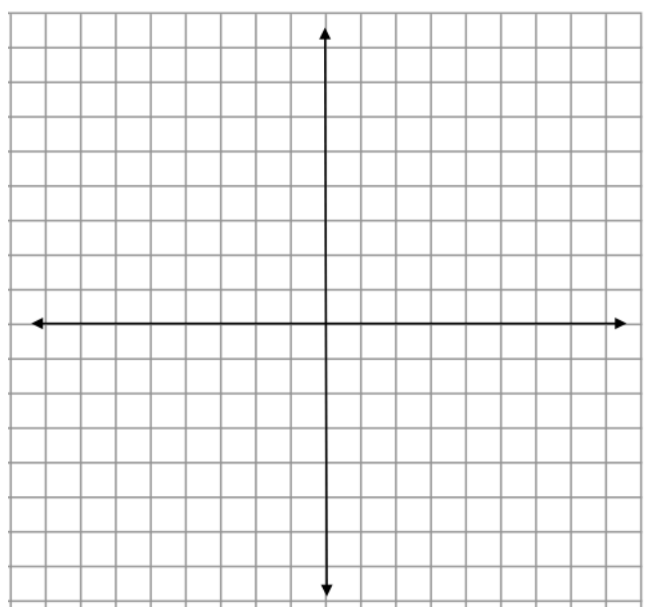
In the direction along the line $y = -3x$, the transformation stretches all points by a factor of two.

In the direction along the line $y = x$, the transformation keeps all points fixed.



1. Use the space on the right to sketch what should happen to the image shown on the left when it is stretched according to the transformation described above. You may use a combination of intuition or calculations, as well as any additional sketches below or on your group's whiteboard.



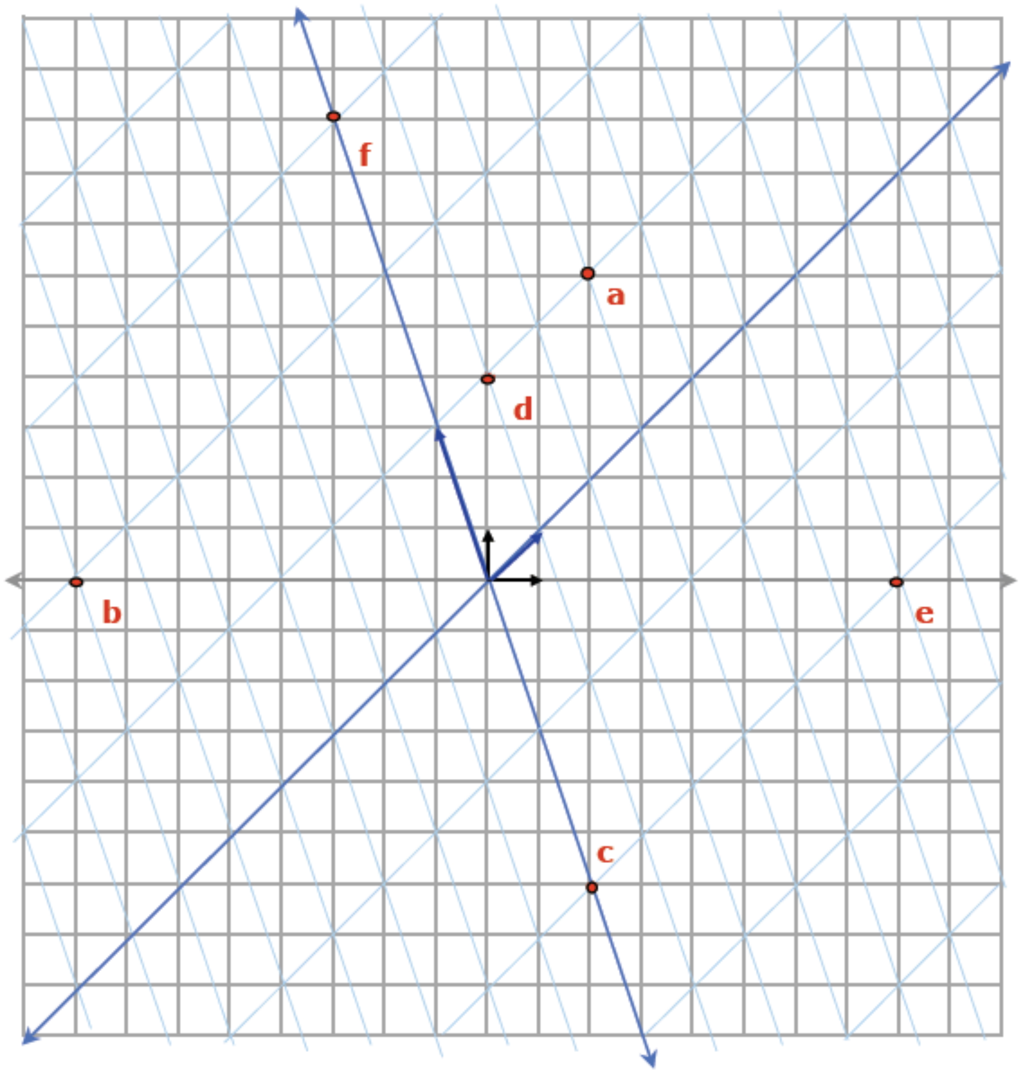


2. Determine what will happen to $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and to $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ under this transformation. Use an initial estimate from your sketch in problem 1. Then try to do a calculation that will determine these locations more precisely.

3. Determine a matrix that allows you to calculate what happens under the transformation to any point on the plane. Use it to check your sketch or improve its accuracy.

The Blue and Black Task

Consider the following two coordinate systems of $\mathbb{R}^2$: the black coordinate system and the blue coordinate system.



1. Write the coordinates of each of the above points relative to both the blue and the black coordinate systems.

2. Determine a matrix that will:

   a. Rename points from the blue coordinate system as points in the black one.

   b. Rename points from the black coordinate system as points in the blue one.

3. Recall the linear transformation from Task 1: vectors along the line $y = -3x$ get stretched by a factor of 2, and vectors along the line $y = x$ remain fixed.

   Determine what happens to each of the vectors below under the transformation. Express the result in both the blue and the black coordinate systems. Describe your methods both graphically and with matrix equations.

   a. $[\boldsymbol{x_1}]_{blue} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

   b. $[\boldsymbol{x_2}]_{blue} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

   c. $[\boldsymbol{x_3}]_{black} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}$

Stretch Factors and Directions

1. The transformation defined by the matrix $A = \begin{bmatrix} 1 & -8 \\ -4 & 5 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in the directions $y = \frac{1}{2}x$ and $y = -x$. Figure out the factor by which anything in the $y = \frac{1}{2}x$ direction is stretched and the factor by which anything in the $y = -x$ direction is stretched.

2. The transformation defined by the matrix $B = \begin{bmatrix} -8 & 2 \\ -55 & 13 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in one direction by a factor of 3 and some other direction by a factor of 2. Figure out what direction gets stretched by a factor of 3 and what direction gets stretched by a factor of 2.

3. The transformation defined by the matrix $C = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$ stretches images in $\mathbb{R}^2$ in two directions. Find the directions and the factors by which it stretches in those directions.

Eigenvalues and Eigenvectors for a 3x3

Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{x}) = A\boldsymbol{x}$ and $A = \begin{bmatrix} 5 & 4 & -6 \\ 3 & 6 & -6 \\ 3 & 4 & -4 \end{bmatrix}$.

Answer the following questions regarding the transformation. Show your work.

1. If you know that the line $c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a stretch direction for the transformation T, what is the stretch factor (eigenvalue) associated with this stretch direction?

2. Given that 2 is a stretch factor (eigenvalue) for the transformation T, determine the set of all stretch directions (eigenvectors) associated with a stretch of 2.

3. From your work on questions 1 and 2, you know two different eigenvalues of the transformation T. Are there any others? If so, find (at least) one. If not, show there can't be any more.