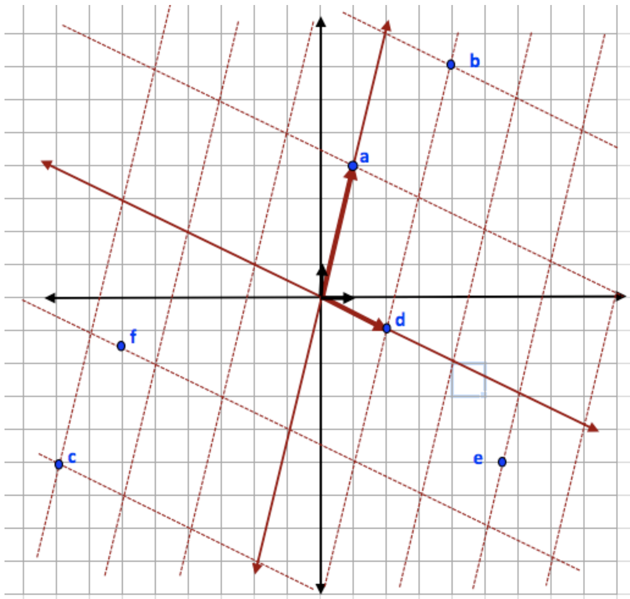
MTH 266, IOLA Homework, Unit 3 Name ____

Instructions: Answer the following questions and attach your answers to this page for submission.

1. Consider the following plane with the both the red and black coordinate systems. The red axes correspond with what are normally thought of as the lines $y = -\frac{1}{2}x$ and $y = 4x$. Let the two red vectors highlighted along the red axis be the basis vectors for a new "red basis," $\boldsymbol{r} = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$.



   a. Write the coordinates of each of the above points relative to both the red basis $\boldsymbol{r}$ and the black basis $\boldsymbol{\alpha}$.
   b. Determine a matrix that will:
      i. Rename points from the red basis as points in the black one.
      ii. Rename points from the black basis as points in the red one.
   c. Consider now the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that a stretch factor of 2 corresponds to the line $y = -\frac{1}{2}x$and a stretch factor of -1 corresponds to the line $y = 4x$.
      i. Determine what happens to the vector $[\boldsymbol{x}]_r = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ under this transformation and represent the answer according to the black basis $\boldsymbol{\alpha}$. Explain all steps in your solution approach.
      ii. Determine what happens to the vector $[x]_\alpha = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ under this transformation and represent the answer according to the red basis. Explain all steps in your solution approach.
   d. Consider the following diagram, where y is the image of $\boldsymbol{x}$ under the transformation T, the matrices A and D are the stretch matrices in black and red respectively, and P renames vectors from red to black:

$$\begin{array}{ccc} [x]_\alpha & \xrightarrow{A} & A[x]_\alpha = [y]_\alpha \\ P \uparrow & & \uparrow P \\ [x]_r & \xrightarrow[D]{} & D[x]_r = [y]_r \end{array}$$

i. Find the matrices A, D, and P that correspond to this diagram for this particular problem. For each of these, give at least one sentence explaining how you found them.

ii. Pick either question 1ci or question 1cii above and explain, using the above diagram, how there are two different methods in which you could have found the answer to that question.

2. In class we discussed how we could think of the modes of transportation in Gauss' Cabin as a basis for $\mathbb{R}^2$. The travel of the hoverboard was given by $\begin{bmatrix}3\\1\end{bmatrix}$ and that of the magic carpet by $\begin{bmatrix}1\\2\end{bmatrix}$. We found, for instance, that Gauss's cabin (located 107 miles East and 64 miles North of your home) could be reached by riding the hover board forward for 30 hours and the magic carpet forward for 17 hours. Thus, Gauss' location $\boldsymbol{x}$ could be described in two ways: using the standard basis (call it $\boldsymbol{\alpha}$), Gauss lives at $[x]_\alpha = \begin{bmatrix}107\\64\end{bmatrix}$. Using the modes of transportation as a basis, call it $\boldsymbol{\beta} = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$, Gauss lives at $[x]_\beta = \begin{bmatrix}30\\17\end{bmatrix}$.

   a. Suppose Uncle Cramer's house $\boldsymbol{w}$ is located at $[\boldsymbol{w}]_\alpha = \begin{bmatrix}25\\17\end{bmatrix}$. Describe this location as a vector in the "travel" coordinate system $\boldsymbol{\beta}$. Explain.

   b. Suppose you visit a museum $\boldsymbol{v}$ located at $[\boldsymbol{v}]_\beta = \begin{bmatrix}8\\3\end{bmatrix}$. Describe the location of the museum as a vector in the standard coordinate system $\boldsymbol{\alpha}$. Explain.

   c. Express each of the following in terms of the travel basis $\boldsymbol{\beta}$: $[\boldsymbol{u}_1]_\alpha = \begin{bmatrix}7\\-1\end{bmatrix}$, and $[\boldsymbol{u}_2]_\alpha = \begin{bmatrix}-6\\3\end{bmatrix}$.

   d. Express each of the following in terms of the standard basis $\boldsymbol{\alpha}$: $[\boldsymbol{z}_1]_\beta = \begin{bmatrix}7\\-1\end{bmatrix}$, and $[\boldsymbol{z}_2]_\alpha = \begin{bmatrix}-6\\3\end{bmatrix}$.

3. Find the eigenvectors and eigenvalues of the matrix $B = \begin{bmatrix}1 & 0 & 1\\0 & 4 & 0\\12 & 2 & 2\end{bmatrix}$. Write 1-3 sentences that interpret these geometrically—in other words, what do the eigenvectors and eigenvalues tell you about the transformation geometrically?

4. A transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ stretches images by $\frac{1}{4}$ in the direction of $\begin{bmatrix}-1\\0\\2\end{bmatrix}$ and stretches images by -3 in the directions of both $\begin{bmatrix}0\\-1\\5\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\9\end{bmatrix}$.

   a. Under this transformation, the vector $\begin{bmatrix}3\\-5\\37\end{bmatrix}$ stretches by -3. Explain why this is true.

   b. Without finding the matrix A, describe what happens to the following vectors under the transformation T. Be sure to justify your conclusions.

      i. $\begin{bmatrix}3\\0\\6\end{bmatrix}$ ii. $\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{9}{2}\end{bmatrix}$ iii. $\begin{bmatrix}-1\\-2\\6\end{bmatrix}$ iv. $\begin{bmatrix}-1\\0\\0\end{bmatrix}$

5. Consider a 2×2 matrix A that has an eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with associated eigenvalue of -3, and an eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with associated eigenvalue of 9. Determine the image of the vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the transformation defined by A in the following two ways:
    a. Set up and solve a system of equations to determine A and then find $A\boldsymbol{x}$.
    b. Write a vector equation that represents $\boldsymbol{x}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then find the image of $\boldsymbol{x}$ by transforming the component parts of the vector equation.
    c. Sketch a graph that uses $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as basis vectors for a new basis for $\mathbb{R}^2$. Determine how $\boldsymbol{x}$ is a linear combination of these vectors (graphically), carry out the transformation by seeing where the vectors go on the graph when transformed, and then state the image of $\boldsymbol{x}$ under the transformation.
    d. Make use of the equation $A\boldsymbol{x} = PDP^{-1}\boldsymbol{x}$ to find the image of $\boldsymbol{x}$ by using the given information to determine P, D, and P^{-1} and then transforming $\boldsymbol{x}$.

6. $A = \begin{bmatrix} 1 & -3 & 3 \\ 2 & -2 & 2 \\ 2 & 0 & 0 \end{bmatrix}$ has characteristic polynomial $det\ A - \lambda I = -\lambda(\lambda + 2)(\lambda - 1)$. For each eigenvalue, find the associated eigenvectors and describe what happens to them geometrically under the transformation defined by A.

7. The matrix A has an eigenvalue of -5 with corresponding 2 eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 5 \\ 2 \end{bmatrix}$, and an eigenvalue of 7 with corresponding eigenvectors $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -1 \\ 2 \\ 3 \end{bmatrix}$. Find the matrix A (round each entry to two decimal places). You may use technology to find your answer, but make sure to explain your reasoning.

8. In class we had the following definition: a matrix A is called diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Since P can be considered to be the matrix with the eigenvectors of A as its columns, we also have the following theorem: An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
    a. Explain in your own words why this theorem is true (Just a short intuitive explanation, not a rigorous proof).
    b. Determine if the matrix A is diagonalizable. If so, find P, D, and P^{-1}. If not, explain why not.

$$A_1 = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

9. Construct examples of each of the following and explain:
    a. A 2×2 matrix that is invertible but not diagonalizable
    b. A 2×2 non-diagonal matrix that is diagonalizable but not invertible