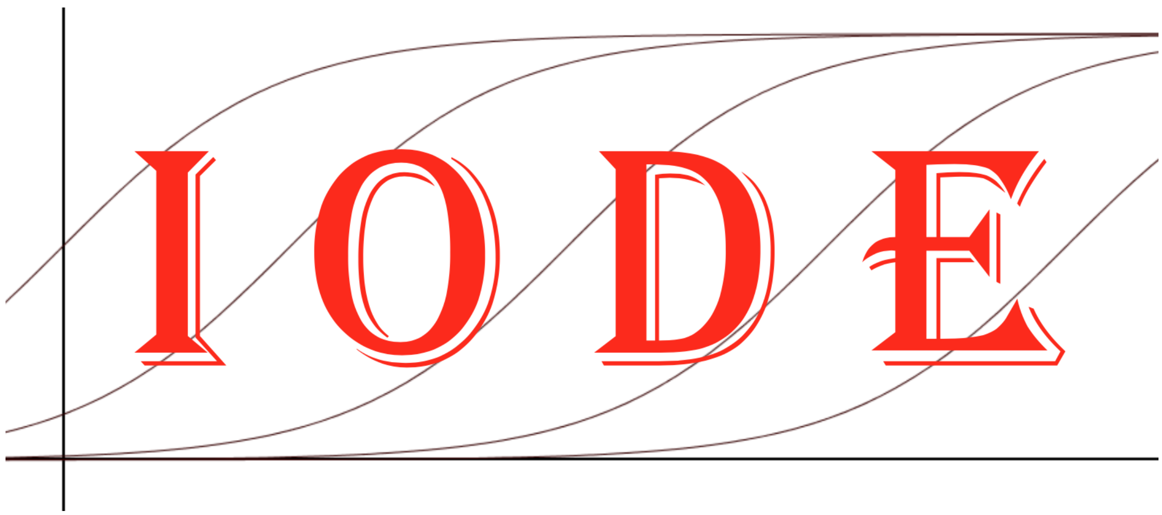
INQUIRY ORIENTED DIFFERENTIAL EQUATIONS



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 https://iode.wordpress.ncsu.edu

Bees and Flowers

Often scientists use rate of change equations in their study of population growth for one or more species. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are *harmed by* interaction) or cooperative (that is, both species *benefit from* interaction).

1. Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

(i) $$\frac{dx}{dt} = -5x + 2xy$$
$$\frac{dy}{dt} = -4y + 3xy$$

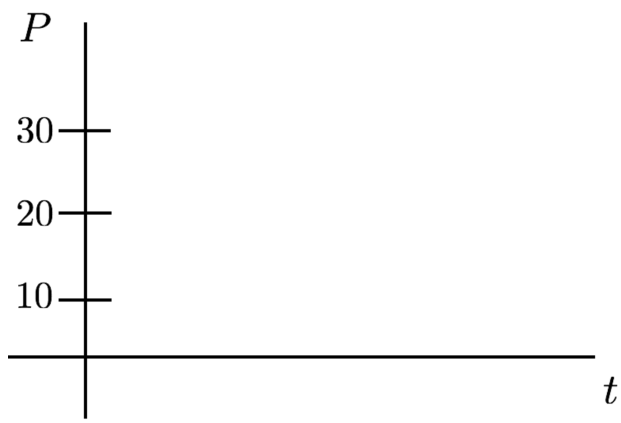
(ii) $$\frac{dx}{dt} = 4x - 2xy$$
$$\frac{dy}{dt} = 2y - xy$$

A Simplified Situation

The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species (*e.g.*, fish)
- The species has been in its habitat (*e.g.*, a lake) for some time prior to what we call $t = 0$
- The species has access to unlimited resources (*e.g.*, food, space, water)
- The species reproduces continuously

2. Given these assumptions for a certain lake containing fish, sketch three possible population versus time graphs: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.



   (a) For your graph starting with $P = 10$, how does the slope vary as time increases? Explain.

   (b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

3. This situation can also be modeled with a rate of change equation, $\frac{dP}{dt} = \textit{something}$. What should the "something" be? Should the rate of change be stated in terms of just P, just t, or both P and t? Make a conjecture about the right hand side of the rate of change equation and provide reasons for your conjecture.

What Exactly is a Differential Equation and What are Solutions?

A **differential equation** is an equation that relates an unknown function to its derivative(s). Suppose $y = y(t)$ is some unknown function, then a differential equation, or rate of change equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. For example, all of the following are differential equations.

$$\frac{dP}{dt} = kP, \qquad \frac{dy}{dt} = y + 2t, \qquad \frac{dy}{dt} = t^2 + 5, \qquad \frac{dy}{dt} = \frac{6y-2}{ty}, \qquad \frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$$

In particular, these are all examples of *first order* differential equations because only the first derivative appears in the equation. Given a rate of change equation for some unknown function, **solutions** to this rate of change equation are *functions* that satisfies the rate change equation. A constant function that satisfies the differential equation is called an **equilibrium solution**.

One way to read the differential equation $\frac{dy}{dt} = y + 2t$ aloud you would say, "dee y dee t equals y plus two times t." However, this does **not** relate to the *meaning* of the solution. How might you read this differential equation *with meaning*?

4. (a) Is the function $y = 1 + t$ a solution to the differential equation $\dfrac{dy}{dt} = \dfrac{y^2-1}{t^2+2t}$? How about the function $y = 1 + 2t$? How about $y = 1$? Explain your reasoning.

   (b) Is the function $y = t^3 + 2t$ a solution to the differential equation $\dfrac{dy}{dt} = 3y^2 + 2$? Why or why not?

5. Figure out all the functions that satisfy the rate of change equation $\dfrac{dP}{dt} = 0.3P$.
   (*Hint*: read the differential equation with meaning.)

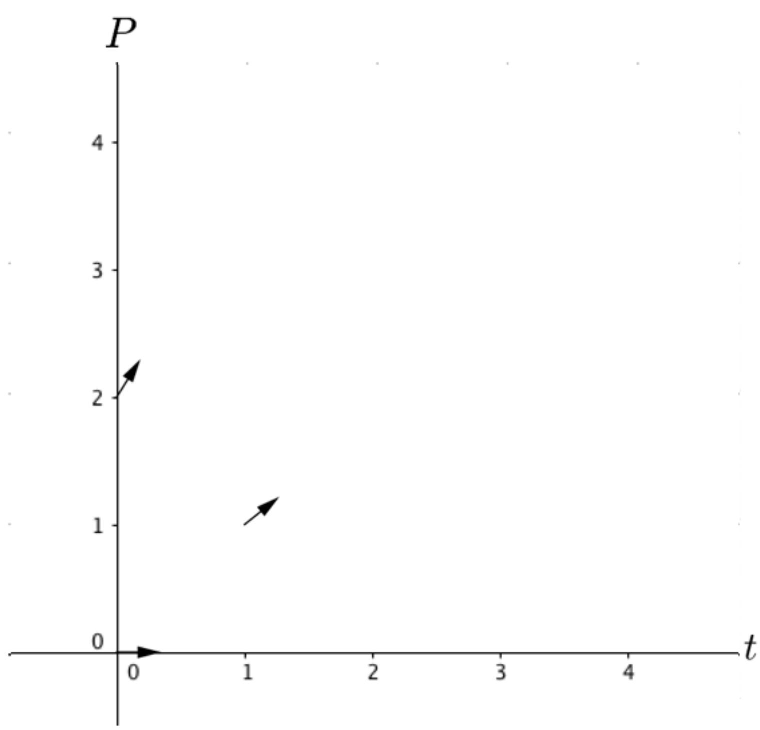
6. Figure out all of the solutions to the differential equation $\dfrac{dy}{dt} = t^2 + 5$.

Slope Fields

A **slope field** is a graphical representation of a rate of change equation. Given a rate of change equation, if we plug in particular values of (t, y) then $\frac{dy}{dt}$ tells you the slope of the tangent vector to the solution at that point.

For example, consider the rate of change equation $\frac{dy}{dt} = y + 2t$. At the point (1, 3), the value of $\frac{dy}{dt}$ is 5. Thus, the slope field for this equation would show a vector at the point (1, 3) with slope 5. A slope field depicts the exact slope of many such vectors, where we take each vector to be uniform length. Slope fields are useful because they provide a graphical approach for obtaining qualitatively correct graphs of the functions that satisfy a differential equation.

7. Below is a partially completed slope field for $\frac{dP}{dt} = 0.8P$.
   (a) Plot many more tangent vectors to create a slope field.
   (b) Use your slope field to sketch in qualitatively correct graphs of the solution functions that start at $P = 0, 0.5$, and 2, respectively. Note: the value of P at an initial time (typically $t = 0$) is called an **initial condition**.
   (c) Recall that a solution to a differential equation is a function that satisfies the differential equation. Explain how the graph with initial condition $P(0) = 1$ can graphically be thought of as a solution to the differential equation when the differential equation is represented by its slope field.

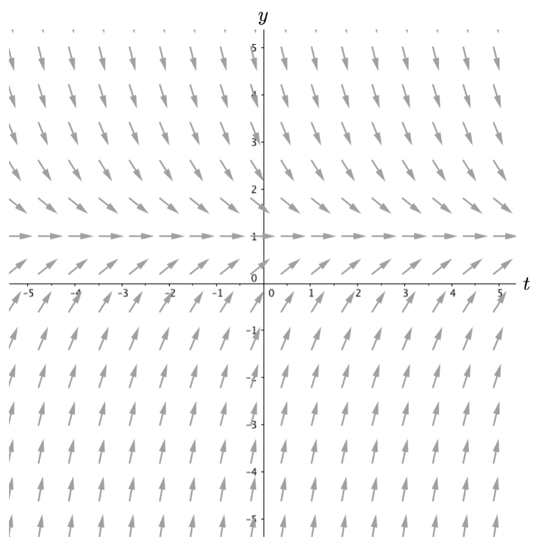


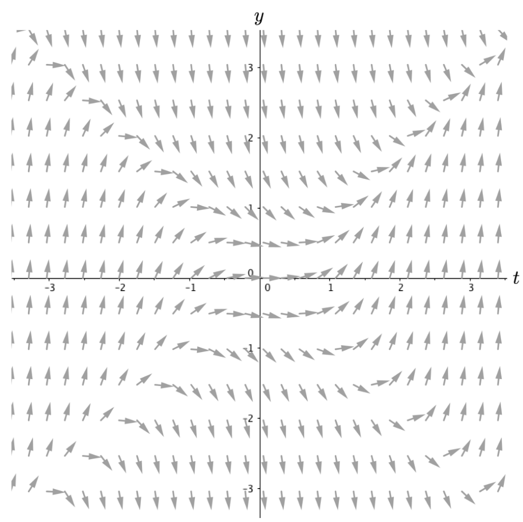
t	P	$\frac{dP}{dt} = 0.8P$
0	0	0
0	2	1.6
1	1	0.8

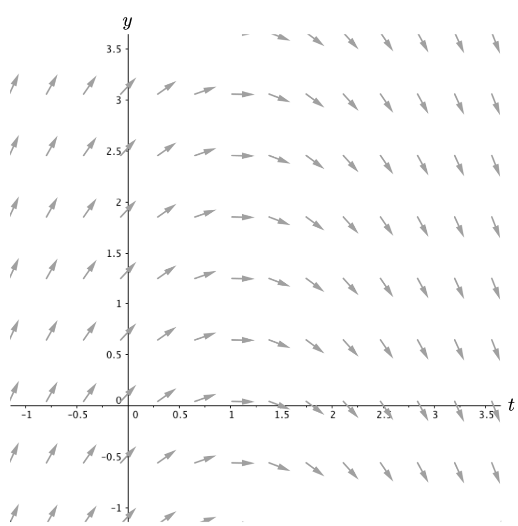
8. Below are seven rate of changes equations and three different slope fields. Without using technology, identify which differential equation is the best match for each slope field (thus you will have four rate of change equations left over). Explain your reasoning.

$$\text{(i) } \frac{dy}{dt} = t - 1 \quad \text{(ii) } \frac{dy}{dt} = 1 - y^2 \quad \text{(iii) } \frac{dy}{dt} = y^2 - t^2 \quad \text{(iv) } \frac{dy}{dt} = 1 - y$$

$$\text{(v) } \frac{dy}{dt} = t^2 - y^2 \quad \text{(vi) } \frac{dy}{dt} = 1 - t \quad \text{(vii) } \frac{dy}{dt} = 9t^2 - y^2$$

(a) 

(b) 

(c) 

9. For each of the slope fields in the previous problem, sketch in graphs of several different qualitatively correct solutions.

Homework Set 1

1. Consider the following systems of rate of change equations:

System A

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{10}\right) - 20xy$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

System B

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$
$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{17}\right) + 25xy$$

In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

Figure out which system is which and explain the reasoning behind your decision.

2. Consider the rate of change equation

$$\frac{dy}{dt} = 0.5y(2+y)(y-8),$$

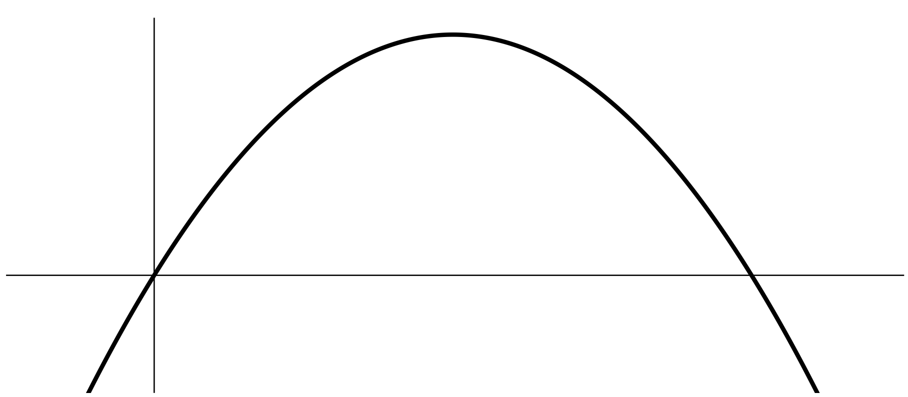
which has been created to provide predictions about the future population of rabbits over time.

(a) For what values of y is $y(t)$ increasing? Explain your reasoning.

(b) For what values of y is $y(t)$ decreasing? Explain your reasoning.

(c) For what values of y is $\frac{dy}{dt}$ neither positive nor negative? What does this imply about the solution function $y(t)$?

3. Valeria created the following graph to help her analyze solutions to the differential equation $\frac{dy}{dt} = 2y\left(1 - \frac{y}{10}\right)$. What is this a graph of (*i.e.*, what are the axes for this graph)? What information about solutions can you glean from this graph?



4. Suppose two students are memorizing the elements on a list according to the rate of change equation

$$\frac{dL}{dt} = 0.5(1 - L),$$

where L represents the fraction of the list that is memorized at any time t.

   (a) If one of the students knows one-third of the list at time $t = 0$ and the other student knows none of the list, which student is learning most rapidly at this instant? Why?

   (b) What does the rate of change equation predict for someone who begins with the list completely memorized? Explain.

   (c) Suppose now that the list is infinitely long, like the decimal representation for π. In reality no one can memorize all the digits to π, but what does the rate of change equation predict will happen for a person who starts out not knowing any of the digits? That is, according to the rate of change equation, if $L = 0$ at time $t = 0$, is there ever a value of t for which $L = 1$? Explain.

5. The letter y appears in two places in the differential equation $\dfrac{dy}{dt} = 0.3y$. Is it appropriate to think of both occurrences of y as function of t? Explain.

6. In algebra, the goal of solving an equation such as $x^2 + 4x = 2$ is to find the values of x that make a true statement. In differential equations, what is the goal of solving an equation such as $\dfrac{dx}{dt} + 4x = 2$?

7. For the differential equation $\dfrac{dy}{dt} = 1 - y^2$,

   (a) Sketch a slope field by hand.

   (b) Describe any shortcuts or patterns you used to make the task easier.

   (c) Sketch several $y(t)$ graphs.

8. Differential equations are often referred to as mathematical models. Explain what the phrase "mathematical model" means to you, what previous experiences you have had with mathematical models, and how the mathematical use of the word model is similar to and/or different from the everyday use of the word model (*e.g.*, fashion model, model airplane, model student).

A Rate of Change Equation for Limited Resources

In a previous problem we saw that the rate of change equation $\frac{dP}{dt} = 0.3P$ can be used to model a situation where there is one species, continuous reproduction, and unlimited resources. In most situations, however, the resources are not unlimited, so to improve the model one has to modify the rate of change equation $\frac{dP}{dt} = 0.3P$ to account for the fact that resources are limited.

1. (a) In what ways does the modified rate of change equation

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right)$$

account for limited resources? (Think of 10 as scaled to mean 10,000 or 100,000)

(b) How do you interpret the solution with initial condition $P(0) = 10$?

(c) Open the Slope Field Viewer, https://ggbm.at/ZGeeGQbp, and plot the slope field for

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right).$$

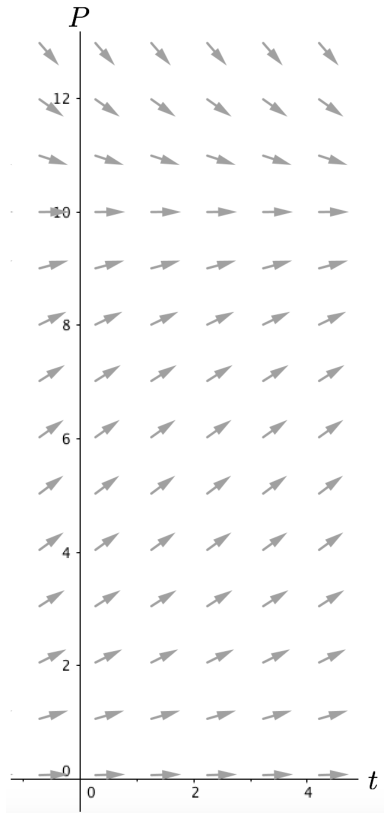
(Note: In the Slope Field Viewer you will need to use the variable y instead of P, and you may want to change the viewing window using the button on the right of the applet.) In what ways are your responses to parts 1a and 1b visible in the slope field?

(d) In this problem, negative P values do not make sense, but we can still mathematically make sense of the slope field for negative P values. Explain why the slope field looks the way it does below the t-axis.

2. If there are initially $P(0) = 2$ fish in the lake, approximately how many fish are in the lake at time $t = 2$? How did you arrive at your approximation? (Hint: Initially $\frac{dP}{dt} = 0.48$, but what meaning does 0.48 have?)

Using a Slope Field to Predict Future Fish Populations

Below is a slope field for the rate of change equation $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right)$.



3. (a) On the slope field above, stitch together in a tip to tail manner several tangent vectors to produce a graph of the population versus time if at time $t = 0$ we know there are 8 fish in the lake (again, think of 8 as scaled for say, 8000 or 80,000).

   (b) Reproduce your technique as much as possible using the Slope Field Stitcher applet, https://ggbm.at/FZn4WHeU. You can use the arrow buttons to move the initial vector around, and then create subsequent vectors to stitch on using the appropriate button.

4. Explain how you are thinking about rate of change **in your method**. For example, is the rate of change constant over some increment? If yes, over what increment? If no, is the rate of change always changing?

5. Using the differential equation $\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$ and initial condition $P(0) = 10$, José and Julie started the following table to numerically keep track of their tip-to-tail method for connecting tangent vectors. Explain José's and Julie's approach and complete their table. **Round to two decimal places.**

t	P	$\frac{dP}{dt}$
0	10	5
0.5	12.5	
1.0		
1.5		

6. Using the same differential equation and initial condition as José and Julie, Derrick and Delores started their table as shown below. Explain how Derrick and Delores' approach is different from José and Julie's and then complete their table. **Round to two decimal places.**

$$\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$$

t	P	$\frac{dP}{dt}$
0	10	5
.25	11.25	
.5		
.75		

7. Which approach do you think is more accurate and why?

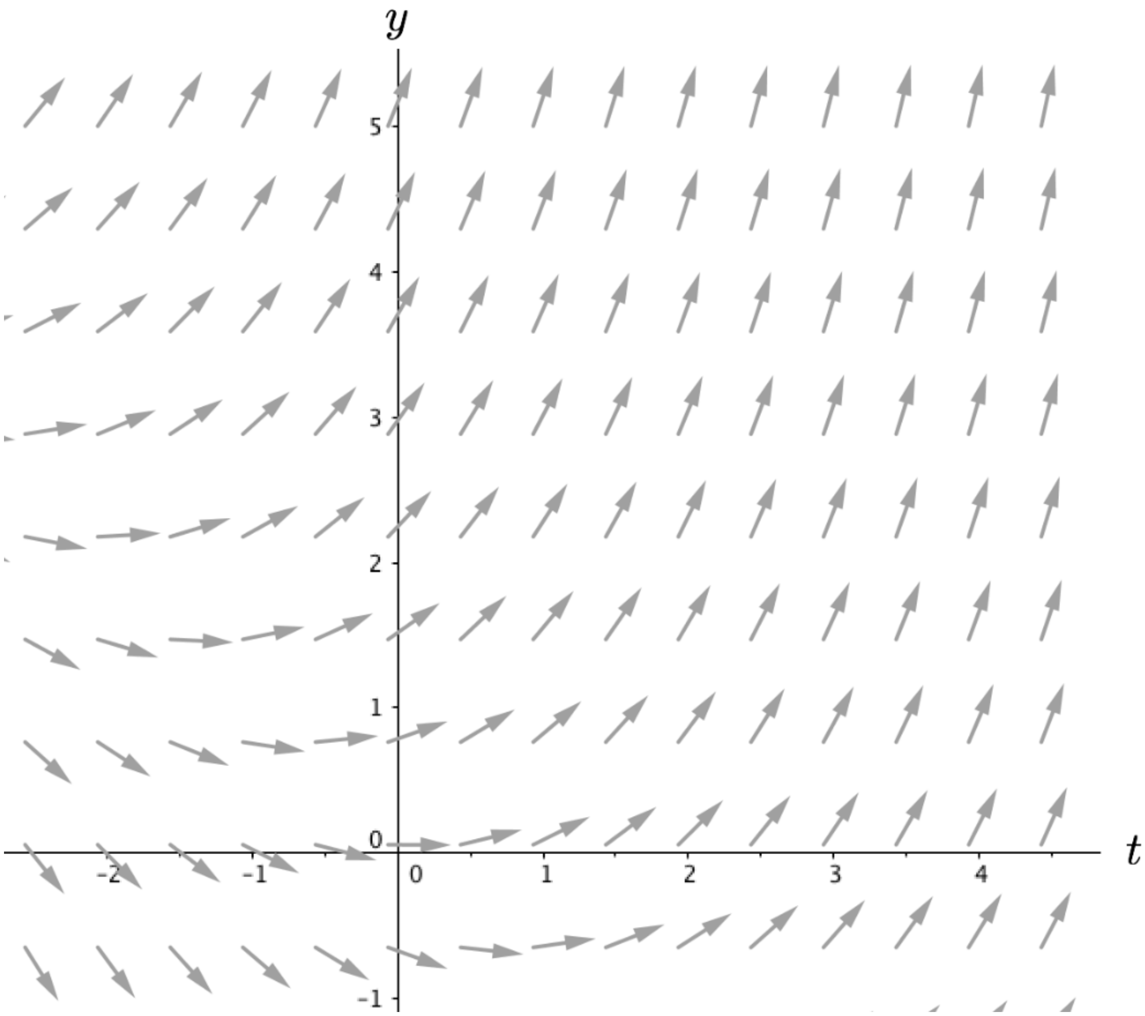
8. (a) Consider the differential equation $\frac{dy}{dt} = y + t$ and initial condition $y(0) = 4$. Use José and Julie's approach to find $y(1.5)$. Show your work graphically and in a table of values.

   (b) Is your value for $y(1.5)$ the exact value or an approximate value? Explain.

9. **Generalizing your tip-to-tail approach**. Create an equation-based procedure/algorithm that would allow you to predict future y-values for any differential equation $\frac{dy}{dt}$, any given initial condition, and any time increment.

Homework Set 2

1. A slope field for the differential equation $\dfrac{dy}{dt} = 0.5(y + t)$ is shown below.

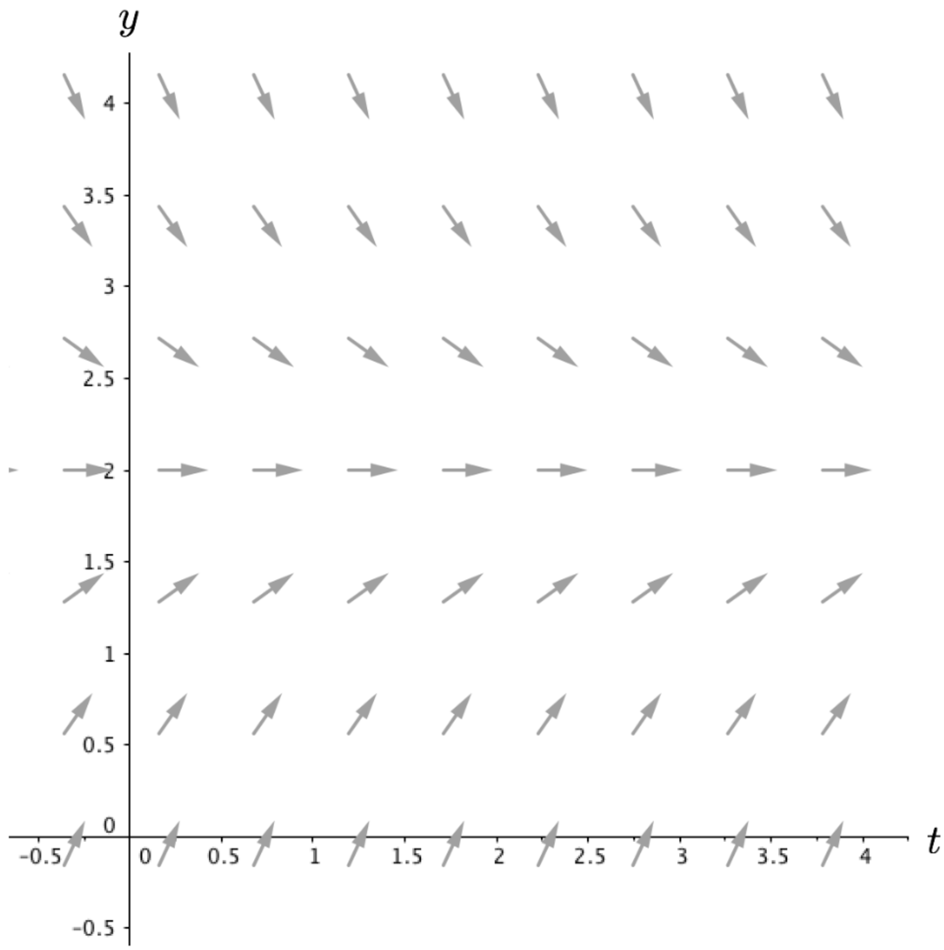


   (a) For the initial condition of $y(0) = 1$, sketch on the above slope field what you think two iterations of the "tip-to-tail" method with a step size of 1 unit should look like. Do this **without** doing any computations.

   (b) Again, without doing any computations, sketch on the same slope field what you think three iterations with a step size of 0.5 units should look like for the same initial condition (perhaps using a different color).

   (c) Use the tip-to-tail (*i.e.*, Euler's) method to numerically compute approximations for parts 1a and 1b and then compare your graphical predictions to the numerical results.

2. Consider a differential equation with the given slope field and the initial value $y(0) = 1$.



(a) Explain why, if you wanted to approximate $y(2)$ using two steps of Euler's method, you would need $\Delta t = 1$.

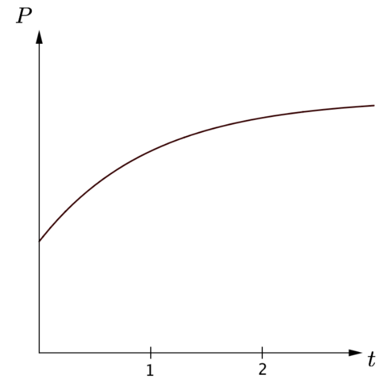
(b) Use a straight edge to graph two steps of Euler's method to approximate $y(2)$.

(c) This time, instead of using two steps of Euler's method, sketch on the same slope field what it would look like if you used four steps of Euler's method to approximate $y(2)$.

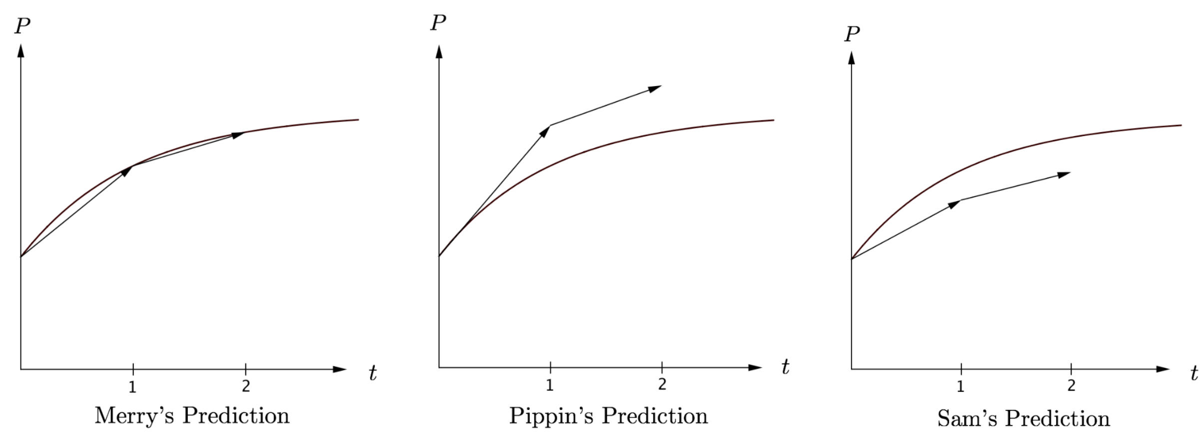
(d) Besides the obvious difference that the step size is different, state two other things that are different between your answers to parts 2b and 2c.

(e) Besides the obvious fact that they both use Euler's method, what is similar about the first step to your answers to parts 2b and 2c?

3. Suppose we have a rate of change equation and initial condition for the population of raccoons in Lake County. Below is a graph of an **exact** solution.



Merry, Pippin, and Sam attempted to use the "tip-to-tail" Euler method to predict what the population of raccoons would be at time $t = 2$, with time increments one unit. However, they arrived at different graphs for their predictions. Their predictions are given below, and are shown with the exact solution.



For each prediction, give reasons as to whether or not each person illustrated the correct relationship between Euler's method and the exact solution.

4. Suppose the function $y(t) = 6t + 1$ is a solution to a particular differential equation. For the initial condition $y(0) = 1$, is a graph of the tip-to-tail Euler method exactly the same as the graph of the exact solution? Does your response depend on step size? Explain.

5. Compute by hand four steps of the tip-to-tail Euler method for the differential equation $\dfrac{dy}{dt} = y - t$ with initial condition $y(0) = 2$ and step size 0.5.

6. **Euler's Method Using a Spreadsheet.** Learning to use a spread sheet for various applications in engineering and mathematics is a valuable skill. Your task in this problem is to use Excel to generate as many steps of the Euler method that you want. *If you are already familiar with Excel, skip the example below and go directly to part a.*

   EXAMPLE: Here are step by step instructions for how to use Excel to generate 15 steps of the algorithm $Y_{\text{next}} = 2 \cdot Y_{\text{now}} + 1$ with initial condition $Y = 3$.

   - Open an Excel workbook
   - Select cell A1 by clicking on the cell in this location and type in Ynow as a column heading
   - Select cell B1 and create a column heading called Ynext
   - Select cell A2 and type in the number 3 (this is the given initial Y-value)
   - Select cell B2 and type =2*A2+1 (after pressing Enter the number 7 will appear in this cell)
   - Select cell A3 and type =B2
   - Select and copy cell B2 (An animated dashed-line will appear around the cell)
   - Select cells B3 through B15 and paste
   - Select and copy cell A3
   - Select cells A4 through A15 and paste
   - Do a few hand computations to verify the results

   (a) Using a step size Δt of your choice, figure out how to use Excel to generate at least 20 steps for Euler's method, $y_{\text{next}} = y_{\text{now}} + (\frac{dy}{dt})_{now} \cdot \Delta t$, for the differential equation $\frac{dy}{dt} = 0.3y(1 - \frac{y}{12.5})$ with initial condition $y(0) = 3$. In order to make it easier to graph the results, make your first column t_{now} and your second column y_{now}. Turn in a print out your results and verify the first three steps by hand.

   (b) Use the Chart Wizard scatter plot option to create a graph of your (t, y) data from part 6a. An easy way to do this is to first highlight all the data in the t_{now} and y_{now} columns, select Chart Wizard, and follow the prompts. Turn in a print out of your results.

7. Two students are having a discussion about the equal sign in the rate of change equation $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{100}\right)$. One student says he thinks about the equal sign as instructions for calculating. The other student says he thinks about the equal sign as a kind of mirror. How do you think about the equal sign in a rate of change equation?

8. A group of scientists created the differential equation $\dfrac{dP}{dt} = 0.8P\left(1 - \dfrac{P}{5}\right)$ to predict future fish populations in Lake Minnetonka, where P represents thousands of fish and t is in years.

   (a) If you were to plot a slope field for this rate of change equation, what window for the P and t values would you use to make sure the most important features are clearly shown? Explain.

   (b) What does this rate of change equation predict about the long-term outcome of the fish population if the initial population is 2 (*i.e.*, $P = 2$ at $t = 0$)? How about if $P = 6$ at $t = 0$?

   (c) Why are the predictions you made in part 8b reasonable (or not) for a fish population? Explain.

   (d) Carry out by hand three steps of Euler's method with a step size of 0.5 for the initial condition $P(0) = 5$.

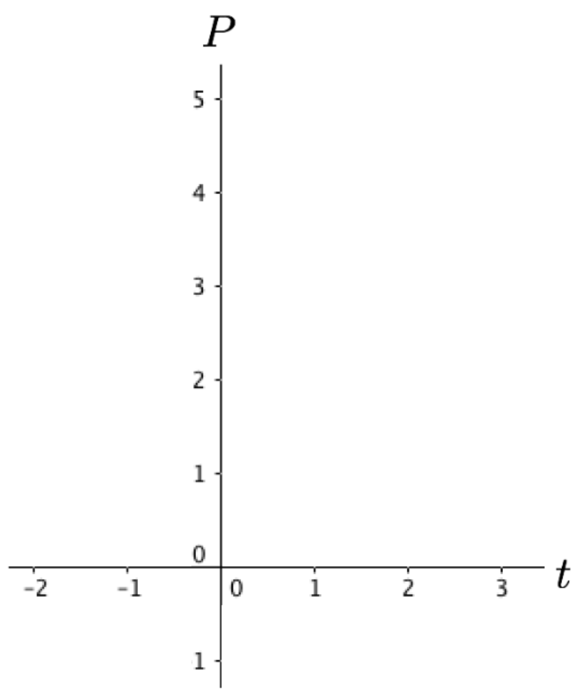
Comparing Predictions

Jerry and Tom are using the differential equation $\dfrac{dP}{dt} = 0.2P$ to make predictions about the number of a particular species of fish in Lake Michigan. They know that the initial population P is 2 at time $t = 0$ (as before, think of 2 as scaled for say, 2,000 or 20,000).

Although Jerry and Tom have the same goal (to obtain predictions for future fish population), they have different approaches to achieve this goal.

- Tom's approach is to create a graph of the number of fish versus time by connecting slope vectors tip-to-tail, where the rate of change is constant over some time interval, for example $\Delta t = 0.5$.
- Jerry's approach is to create a graph of the number of fish versus time by using a continuously changing rate of change.

1. Sketch Tom and Jerry's approaches below. Will these two approaches result in the same predictions for the number of fish in, say, 2.5 years? If yes, why? If not, how and why will the graphs of their approaches be different?



Separation of Variables

2. **Finding the exact solution**. Jerry's approach involves using a continuously changing rate of change, which corresponds to finding an "exact solution."

   (a) Why do you think the phrase "exact solution" is used to describe the result of Jerry's approach? Explain why it is appropriate to describe the result of Tom's approach as an "approximate solution".

   (b) Use the chain rule to write down, symbolically, the derivative with respect to t of $\ln(P)$, where P is shorthand for $P(t)$.

Next you will learn a technique for finding the exact solution corresponding to Jerry's approach. We begin by considering the chain rule.

(c) The following is a method to find the analytic solution to $\frac{dP}{dt} = 0.2P$. For now assume that $P > 0$. This assumption corresponds to the population growth context and it will make the algebra easier and hence the underlying idea clearer.

Divide both sides of $\frac{dP}{dt} = 0.2P$ by P	
Replace $\frac{1}{P}\frac{dP}{dt}$ with $[\ln(P)]'$	
Write integrals with respect to t on both sides	
Apply the Fundamental Theorem of Calculus to integrate both sides	
Solve for P (and remember that P is actually a function, $P(t)$)	
Show that P can be written as $P(t) = ke^{0.2t}$	

The end result, $P(t) = ke^{0.2t}$ is called the **general solution** because it represents all possible functions that satisfy the differential equation. We can use the general solution to find any **particular solution**, which is a solution that corresponds to a given initial condition.

3. Use the same technique to find the general solution to $\frac{dy}{dt} = \frac{t}{3y^2}$. The first step is done for you.

$$3y^2\frac{dy}{dt} = t$$

4. In practice, we often circumvent explicit use of the chain rule and instead use a shortcut to more efficiently find the general solution. The shortcut involves treating the derivative $\frac{dP}{dt}$ as a ratio and "separating" the dP and dt. In the table below, follow the instructions to see how the shortcut works, using again the equation $\dfrac{dP}{dt} = 0.2P$. (See http://kevinboone.net/separation_variables.html) for a nice explanation of the shortcut).

'Separate' the dP from the dt so that dP and P are on the same side. (If there are t's in the equation they must go on the same side as dt.)	
Integrate both sides of the equation (one side with respect to P, the other with respect to t)	
Continue as before to arrive at a solution of the form $P(t) =$ _____	

5. Use the shortcut to find the general solution to $\dfrac{dy}{dt} = \dfrac{t}{3y^2}$.

6. A differential equation together with an initial condition is called an **Initial Value Problem** (IVP). To solve an IVP one first must find the general solution and then use the initial condition to find the particular solution corresponding to the initial condition.
Solve the following IVP:

$$\frac{dy}{dt} = \frac{t}{y} \qquad y(2) = -1$$

(a) For what values of t is your solution valid? Why?

(b) Check to see that your particular solution "fits" the differential equation by substituting the solution and its derivative into the original differential equation.

(c) Use the GeoGebra applet, https://ggbm.at/SbHk2n4H, to check to see that your specific solution "fits" the differential equation by plotting the slope field and then plotting the graph of the solution on top of the slope field. Explain how this relates to Jerry's approach.

(d) Even though $\frac{dy}{dt}$ is undefined when $y = 0$, the solution function can be defined such that $y(2) = 0$. What should the graph of this solution look like in the slope field?

Making Connections

7. For the first slope field for $\frac{dL}{dt} = 0.5(1 - L)$ on the following page,

   (a) Using Jerry's approach, sketch as accurately as possible a graph of the solution with initial condition $L(0) = 1/3$.

   (b) Make a copy of this sketch on a transparency.

   (c) If you wanted to obtain the graph of the solution with initial condition $L(0) = 1/2$, how, if at all, might you move the copy of your graph with initial value 1/3 so that it is now a graph of the solution with initial value 1/2? What feature of the differential equation justifies your approach?

   (d) Find the general solution for $\dfrac{dL}{dt} = 0.5(1 - L)$ and explain how your results from part 7c can be understood from the general solution.

8. For the second slope field for $\frac{dh}{dt} = -t + 1$ on the following page,

   (a) Using Jerry's approach, sketch as accurately as possible a graph of the solution with initial condition $h(0) = 1/2$.

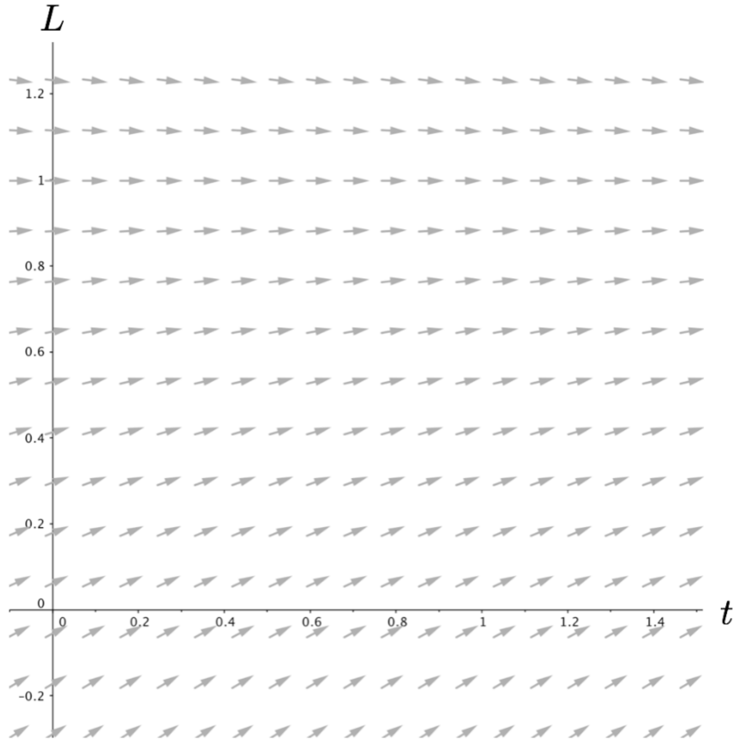
   (b) Make a copy of this sketch on a transparency.

   (c) If you wanted to obtain the graph of the solution with initial condition $h(0) = 1$, how, if at all, might you move the copy of your graph with initial value 1/2 so that it is now a graph of the solution with initial value 1? Explain your idea and provide reasons for why your idea makes sense.

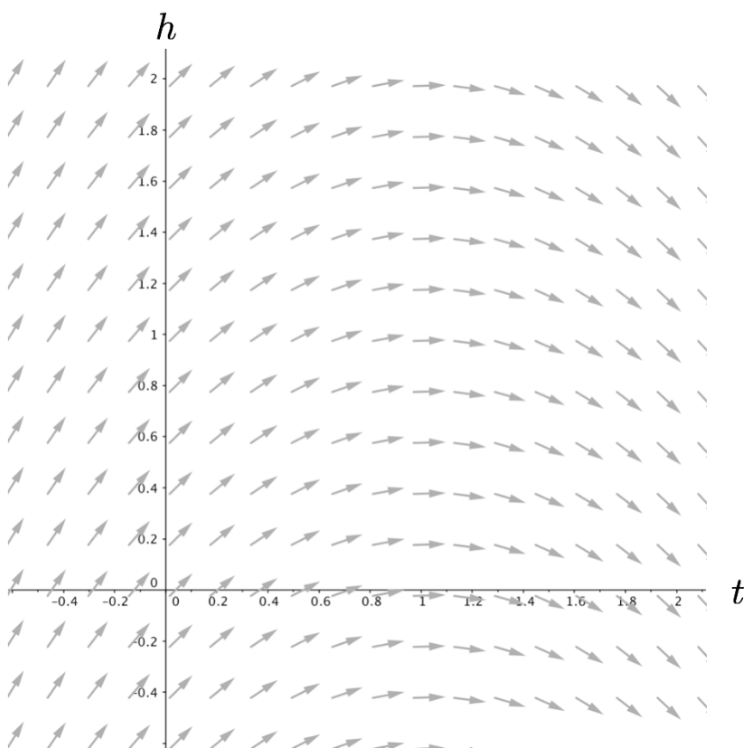
   (d) Find the general solution for $\frac{dh}{dt} = -t + 1$ and explain how your results from part 8c can be understood from the general solution.

9. Give an example of a differential equation where neither of your ideas from 7c and 8c will work and provide reasons for your response.

Slope Field for $\dfrac{dL}{dt} = 0.5(1 - L)$



Slope Field for $\dfrac{dh}{dt} = -t + 1$



Homework Set 3

1. When you solve an equation such as $x^2 - 3 = 1$, you get two numbers $x = 2$ and $x = -2$. When you solve a differential equation, what do you get?

2. Find the general solution to the following differential equations.

   (a) $\dfrac{dy}{dt} = t^4 y$

   (b) $\dfrac{dy}{dt} = 2y + 1$

   (c) $\dfrac{dy}{dt} = t\sqrt[3]{y}$

   (d) $\dfrac{dy}{dt} = \dfrac{t}{y+1}$

   (e) $2\dfrac{dy}{dx} = xy(x+1)$

3. Find the particular solution to the following initial value problems.

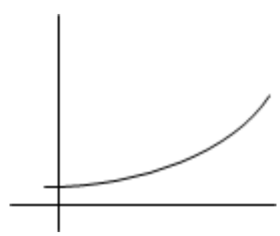
   (a) $\dfrac{dy}{dt} = \dfrac{-t}{y}, \qquad y(0) = 4$

   (b) $\dfrac{dy}{dt} = -\sqrt[3]{y}, \qquad y(0) = 27$

   (c) $\dfrac{dy}{dx} = \dfrac{x(y-2)}{x^2+4}, \qquad y(1) = 5$

4. Develop a differential equation where $y(t) = 6$ is a solution function but $y(t) = 8$ is not a solution function. Explain why your differential equation meets both of these criteria.

5. Denise has created the following graph to go along with the rate of change equation $\frac{dP}{dt} = 0.2P$.



   What is this a graph of? Label the axes and explain your reasoning.

6. Cornelia is working with the differential equation $\dfrac{dy}{dt} = y - t$. She has no method like separation of variables to use but still needs to a way to figure out which, if any, of the following functions are solutions to $\dfrac{dy}{dt} = y - t$.

   (i) $y(t) = t + 2$ (ii) $y(t) = e^t - 1$ (iii) $y(t) = e^t + t + 1$ (iv) $y(t) = t$

   (a) Read the differential equation $\dfrac{dy}{dt} = y - t$ with *meaning*. Write down exactly how you would read the equation with meaning. Recall *reading with meaning* was discussed in Unit 1.

   (b) Explain how Cornelia can use a slope field to determine which, if any, of these functions are solutions to the differential equation, $\dfrac{dy}{dt} = y - t$.

   (c) Use what it means to be a solution to a differential equation to determine which, if any, of these functions are solutions to $\dfrac{dy}{dt} = y - t$. Show all work.

A Salty Tank

1. A very large tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at a rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute.

   (a) Should the rate of change equation for this situation depend just on the amount of salt S in the tank, the time t, or both S and t? Explain your reasoning.

   (b) The following is a general rule of thumb for setting up rate of change equations for situations like this where there is an input and an output:

   $$\text{rate of change} \;=\; \text{rate of change in} \;-\; \text{rate of change out}$$

   Using the above rule of thumb, figure out a rate of change equation for this situation.
   Hint: Think about what the *units* of $\frac{dS}{dt}$ need to be, where S is the amount of salt in the tank in pounds.

(c) Use the slope field for this differential equation in the GeoGebra applet, https://ggbm.at/PFRcbkbZ, to sketch a graph of the solution with initial condition $S(0) = 6$. Reproduce this sketch below. Estimate the amount of salt in the tank after 15 minutes.

The differential equation you developed for the salty tank is not separable, and therefore using the technique of separation of variables is not appropriate. This differential equation is called **first order linear**, which means it has the form

$$\frac{dy}{dt} + g(t) \cdot y = r(t),$$

where $g(t)$ and $r(t)$ are both continuous functions.

The following technique, which we refer to as the **reverse product rule**, can be used find the general solution to a first-order linear equation.

2. Review the product rule as you remember it from calculus. In general symbolic terms, how do you represent the product rule? How would you describe it in words?

Consider the differential equation $\frac{dy}{dt} + 2y = 3$. Note that this is a first order linear differential equation, where $g(t)$ and $r(t)$ are both continuous functions. The following illustrates a technique for finding the general solution to linear differential equations. The inspiration for the technique comes from a creative use of the product rule and the Fundamental Theorem of Calculus, as well as use of the previous technique of separation of variables.

Use the product rule to expand $(yu)'$.	Box 0
In the equation $\frac{dy}{dt} + 2y = 3$, rewrite $\frac{dy}{dt}$ as y'.	Box 1
Notice that the left-hand side of the equation in Box 1 looks a lot like the expanded product rule but is missing the function u. So multiply both sides by u, a function that we will determine shortly.	Box 2
Because, so far, u is an arbitrary function, we can have u satisfy any differential equation that we want. Use $u' = 2u$ to rewrite the left-hand side of Box 2 to look like Box 0.	Box 3

Use separation of variables to solve $u' = 2u$.	Box 4
Replace u in the equation from Box 2 with your solution from Box 4.	Box 5
Show that the equation in Box 5 can be rewritten as $\left(ye^{2t}\right)' = 3e^{2t}$ *Hint*: Consider Box 0.	Box 6
Write integrals with respect to t on both sides. Apply the Fundamental Theorem of Calculus.	Box 7
Obtain an explicit solution by isolating $y(t)$.	Box 8

3. Use the previous technique, which we refer to as the **reverse product rule**, to find the general solution for the Salty Tank differential equation from Problem 1.

4. (a) Use the general solution from problem 3 to find the particular solution corresponding to the initial condition $S(0) = 6$ and then use the particular solution to determine the amount of salt in the tank after 15 minutes. That is, compute $S(15)$. Your answer should be close to your estimate from problem 1c. Is it? If not, you likely made an algebraic error.

   (b) What does your solution predict about the amount of salt in the tank in the long run? How about the concentration?

   (c) Explain how you can make sense of the predictions from 4b by using the differential equation itself.

Homework Set 4

1. Find the general solutions to the following differential equations using separation of variables or the reverse product rule. Give a reason as to why you used the method you chose over the other.

   (a) $\dfrac{dy}{dt} = 2y - t$

   (b) $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2$

   (c) $\dfrac{dy}{dt} = y \sin t$

   (d) $\dfrac{dy}{dt} = \cos t$

2. Solve the following differential equation in two ways: once using separation of variables, and once using the reverse product rule.
$$\frac{dy}{dt} = 2y + 1\,; \quad y(0) = 2$$

3. For each of the following, determine which method(s) could be used to find the general solution. Do NOT actually find the general solutions, just determine any and all techniques that could be used.

   (a) $\dfrac{dy}{dt} = 2y - 3e^{-t}$

   (b) $\dfrac{dy}{dt} = -0.2(75 - y)$

   (c) $\dfrac{dy}{dt} = y^2 + 1$

   (d) $\dfrac{dy}{dt} = e^t y - \cos t$

4. (a) Create a differential equation (different from all those above) that can only be solved with separation of variables.

   (b) Create a differential equation (different from all those above) that can only be solved with the reverse product rule method.

   (c) Create a differential equation (different from all those above) that can be solved with either the reverse product rule method or separation of variables.

5. A tank initially contains 90 lb of salt dissolved in 20 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 3 gal/min, and the mixture flows out of the tank at the same rate. How much salt does the tank contain 6 minutes later?

6. (a) Use our technique for solving linear differential equations to verify that the exact solution to $\dfrac{dy}{dt} = 2y - 3e^{-t}$ with initial condition $y(0) = 1$ is $y(t) = e^{-t}$.
   (b) Compare the long-term behavior of the exact solution and one or more tip to tail Euler's method approximations. Describe and graphically illustrate your results and develop an argument that explains or accounts for these results.
   (c) The Euler algorithm starts at some value, computes the rate of change at that value, and then assumes that this rate of change is going to be constant over a specified time interval. This process gets you to the next value and the entire process is repeated. In other words, for each time interval this recipe uses the rate of change at the beginning of each interval. One idea to improve this process is instead of using the value of the rate of change at the beginning of each time interval, calculate some sort of average of the rate of change values over the time interval and then use that averaged rate of change just as you would in the Euler recipe. Go to the library and look up in one or more differential textbooks the approximation method called Runge-Kutta. Write down what the algorithm for this method and briefly discuss why this method is better than the Euler method.
   (d) Use EXCEL to compare a Runge-Kutta approximation, an Euler's method approximation, and a graph of the exact solution for the differential equation $\dfrac{dy}{dt} = 2y - 3e^{-t}$ with initial condition $y(0) = 1$. Summarize your comparison below and discuss how (and why) the approximations differ from the graph of the exact solution.

7. Thus far in the course, we have approached analysis of rate of change equations in three different ways – analytical approaches, numerical approaches, and graphical approaches. In words understandable to a calculus student planning to take differential equations, describe what it means to analytically, numerically, and graphically analyze solutions to a differential equation. Also, develop an explanation that would help this student understand when you might want to use one approach over the other and what advantages and disadvantages each accompanies each approach.

8. Some textbooks refer to the "reverse product rule" technique as the method of "integrating factors." Do some research using the internet or textbooks and explain how the integrating factor method relates to the reverse product rule.

9. Let's call the salty tank we discussed in class Tank A. Consider the following modifications:

   - Tank B is the same basic scenario as Tank A, but pure water is being pumped into Tank B instead of saltwater.
   - Tank C is the same basic scenario as Tank A, but the rates are switched: saltwater enters Tank C at a rate of 1 gallon per minute, and leaves at a rate of 2 gallons per minute.
   - Tank D is the same basic scenario as Tank A, but Tank D initially contains 6 gallons of pure water.

   (a) Set up and solve initial value problems that correspond to individual Tanks B, C, and D.
   (b) Use a plot to compare four solution curves and discuss how these curves predict/represent outcomes you might expect from the description of each scenario.

10. In this course so far we have discussed various analytic, numerical, and graphical methods and techniques for finding and understanding solutions to differential equations. Below are the methods discussed in this course, up to this point, in the left column. Match the technique to its appropriate category(ies), in the right column.

Technique	Category
Euler's Method	Analytic Technique
Reverse Product Rule	Numerical Technique
Slope Field	Graphical Technique
Separation of Variables	

Proposed Paths of Descent

A group of scientists at the Federal Aviation Association has come up with the following two different rate of change equations to predict the height of a helicopter as it nears the ground:

$$\frac{dh}{dt} = -h \qquad \text{and} \qquad \frac{dh}{dt} = -h^{\frac{1}{3}}$$

For both rate of change equations h is in feet and t is in minutes. The scientists, of course, want their models to predict that a helicopter actually lands - but do either or both of the proposed models predict this?

1. Getting familiar with the differential equations:

   (a) Just by examining the rate of change equations, what can you say about the height of the helicopter as predicted by $\frac{dh}{dt} = -h$ and by $\frac{dh}{dt} = -h^{\frac{1}{3}}$? More specifically, as h approaches zero, what can you say about $\frac{dh}{dt}$ and what does that imply about whether the model predicts that the helicopter lands?

   (b) Sketch your best guess for a height versus time solution graph for each rate of change equation.

2. (a) What do each of the proposed rate of change equations say about the solution to the differential equation if the helicopter is already on the ground? Explain and sketch a corresponding graph of height versus time on the same set of axes from part 1b.

   (b) Interpret the initial condition $h(0) = 0$, and explain why $h(t) = 0$ should be a solution to each differential equation under this initial condition.

3. Use the Geogebra applet, https://ggbm.at/dJsACfAN, to investigate the slope fields. What do the slope fields suggest about whether the model predicts if the helicopters will land? How do the slope fields compare with your sketches from part 1b?

4. Solve the following initial value problems:

   (a) $\dfrac{dh}{dt} = -h$

   (i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

   (b) $\dfrac{dh}{dt} = -h^{\frac{1}{3}}$

   (i) $h(0) = 2$ (ii) $h(0) = 0$ (*Hint*: Use problem 2b)

5. (a) For each differential equation, interpret the results from problem 4 in terms of whether the model predicts the helicopter will ever touch the ground. If so, at what time?

   (b) For each differential equation, interpret the results from problem 4 in terms of whether graphs of (i) and (ii) will ever touch or cross.

6. One difference between the two differential equations is the partial derivative of the right hand side at $h = 0$. That is,
$$\frac{\partial f}{\partial h}, \quad \text{where } f(h) = -h$$
for one differential equation is different than
$$\frac{\partial f}{\partial h}, \quad \text{where } f(h) = -h^{\frac{1}{3}}$$
for the other differential equation.

   Accurately draw graphs of $\dfrac{dh}{dt}$ versus h for both differential equations and use these graphs to determine the partial derivatives at $h = 0$ for each differential equation.

The Uniqueness Theorem

In the formal language of differential equations, the term "unique" or "uniqueness" refers to whether or not two solution functions ever touch or cross each other. Using this terminology, the two solutions you found to $\dfrac{dh}{dt} = -h$ are unique while the two solutions you found to $\dfrac{dh}{dt} = -h^{\frac{1}{3}}$ are not unique. Fortunately, one does not have to always analytically solve a differential equation to determine if solutions will or will not be unique. There is a theorem, the **Uniqueness Theorem**, which sets out conditions for when solutions are unique.

Theorem. Let $f(x, y)$ be a real valued function which is continuous on the rectangle

$$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}.$$

Assume f has a partial derivative with respect to y and that this partial derivative $\partial f/\partial y$ is also continuous on the rectangle R. Then there exists an interval

$$I = [x_0 - h, x_0 + h] \text{ (with } h \leq a)$$

such that the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ defined on the interval I.

7. Explain how the conditions of this theorem relate to solutions of $\dfrac{dh}{dt} = -h$.

8. If you are given a differential equation and determine that the conditions of the uniqueness theorem are NOT met in a specific range of y-values, what can you conclude about the graphs of solution functions within that range of y-values? Explain.

Homework Set 5

1. Suppose two planes start descending at the same time, one is directly above the other and both follow the same differential equation, $\dfrac{dh}{dt} = -h^{1/3}$. Is there any possibility of a midair collision? Will the initially higher one ever get below the initially lower one? Develop two different arguments to support your conclusion, one based on the uniqueness theorem and one based on the fact this differential equation is autonomous and hence graphs of solutions are related to each in a particular way.

2. In light of the **Uniqueness Theorem**, consider the population model
$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{12.5}\right).$$
If $P(0) < 12.5$, will the population ever reach 12.5? Explain.

3. For each differential equation, determine (with reasons) whether or not graphs of solution functions will ever touch any and all equilibrium solution functions (consider both positive and negative values of t).

   (a) $\dfrac{dL}{dt} = .5(1 - L)$ (b) $\dfrac{dy}{dt} = 0.3y\left(1 - \dfrac{y}{10}\right)$ (c) $\dfrac{dy}{dt} = -t + 1$ (d) $\dfrac{dy}{dt} = y^{\frac{1}{2}}$

4. Suppose two students are memorizing a list according to the same model $\dfrac{dL}{dt} = 0.5(1 - L)$ where L represents the fraction of the list that is memorized at any time t. According to the uniqueness theorem, will the student who starts out knowing none of the list ever catch up to the student who knows one-third of the list? Explain.

5. What values of p result in predictions that the helicopter will land in a finite amount of time for the model $\dfrac{dh}{dt} = -h^p$? Explain and show all work.

Analyzing Autonomous DEs: Spotted Owls

A group of biologists are making predictions about the spotted owl population in a forest in the Pacific Northwest. The autonomous differential equation the scientist use to model the spotted owl population is $\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{5}\right)\left(\frac{P}{8} - 1\right)$, where P is in hundreds of owls and t is in years. The problem is that the current number of owls is only approximately known.

1. Suppose the scientists estimate that currently P is about 5 (i.e. there are currently about 500 owls in the forest). Since 5 is only an estimate, they make long-term predictions of the owl population for the initial conditions $P = 4.9$, $P = 5.0$, and $P = 5.1$. *Without using a graphing calculator or other software*, determine the long-term predictions for these initial conditions based on the differential equation. Are they similar or different? That is, will slightly different initial conditions yield only slightly different long-term predictions, or will they be radically different? Carry out a similar analysis if the current number of owls is somewhere around 8.

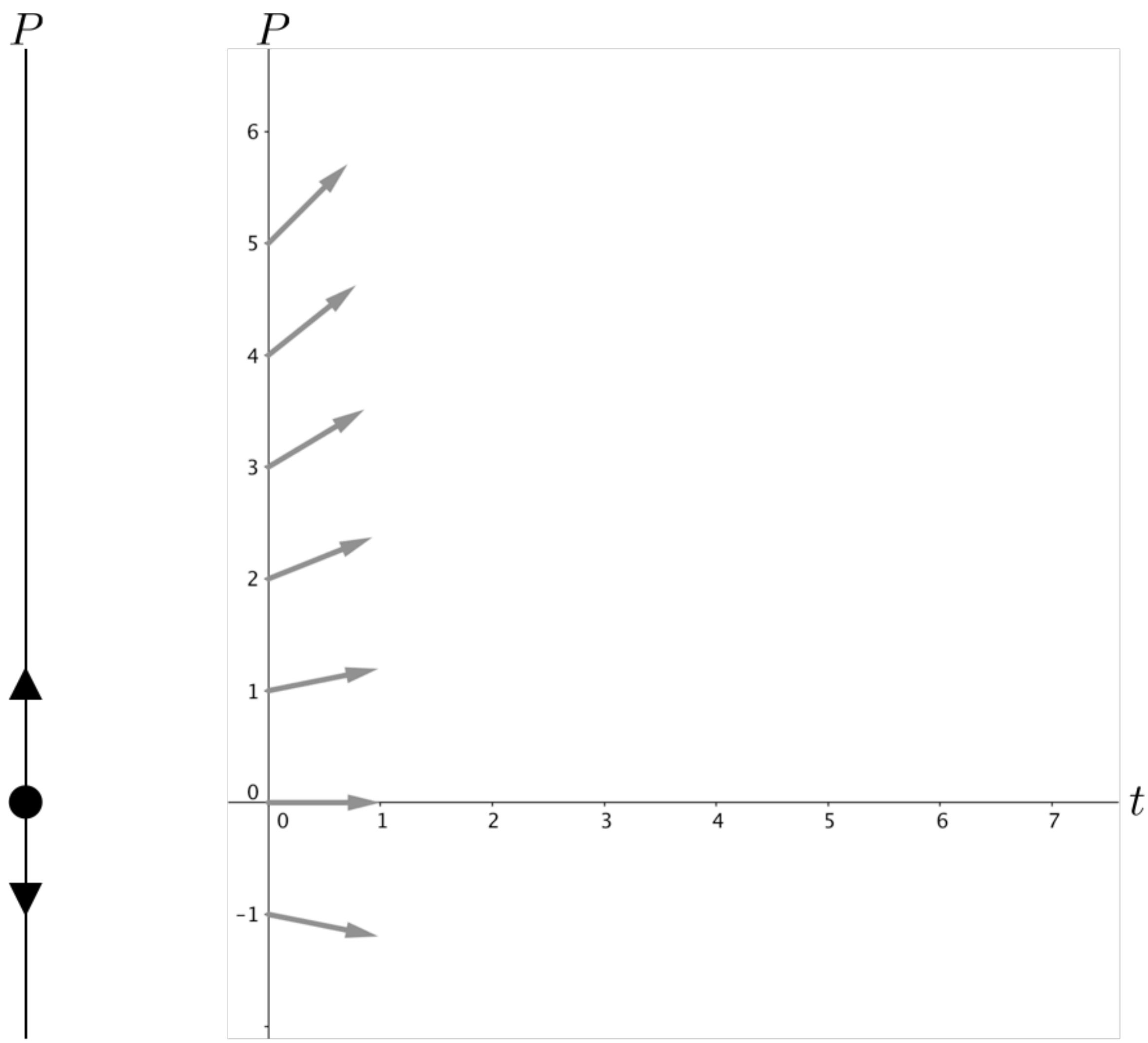
2. Give a one dimensional representation, *without words*, that would describe all solutions to the differential equation.

3. A **phase line** is the standard one-dimensional diagram that depicts the qualitative behavior of solutions to an autonomous differential equation. Label the dots and add arrows to the figure below to represent **all** solutions to the differential equation in Problem 1.

4. For the differential equation in problems 1-3 there are three equilibrium solutions. Recall that equilibrium solutions are constant functions that satisfy the differential equation. How do the other solution functions near each equilibrium solution behave in the long term? If you were to label each of these equilibrium solutions based on the way in which nearby solutions behave, what terms would you use and why?

Phase Lines

5. Dominique is working with the rate of change equation $\frac{dP}{dt} = 0.2P$ and thinks about solutions in terms of whether they are increasing, decreasing, or remaining constant. She illustrates her thinking with the phase line shown below.



(a) Place your fingertip or other small item on the phase line at $P = 0$ and another fingertip or small item at $P = 0$ on the P vs t axes and imagine time moving forward. Explain, with reasons, what happens to your fingertips.

(b) Place your fingertip or other small item on the phase line at $P = 1$ and another fingertip or small item at $P = 1$ on the P vs t axes at (0,1) and imagine time moving forward. Explain, with reasons, what happens to your fingertips.

(c) Place two fingertips or two small items on the phase line, one at $P = 1$ and the other at $P = 3$. What happens to your fingertips as time moves forward? How do your ideas relate to the corresponding P versus t graphs?

(d) Explain how a person could think about the phase line as a one-dimensional projection of all of the two-dimensional $P(t)$ graphs of solutions.

Homework Set 6

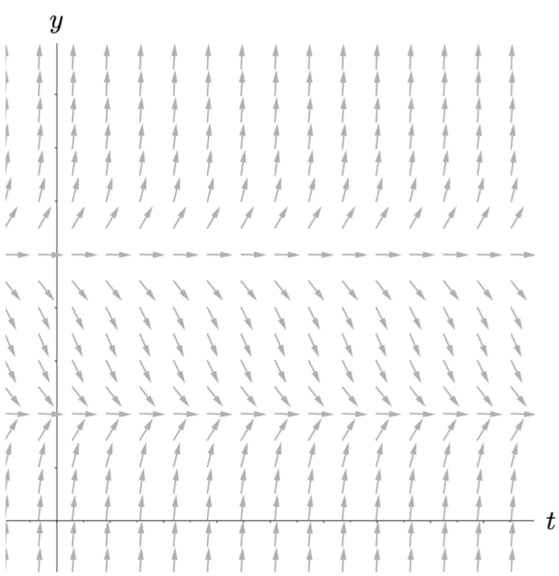
1. For an autonomous differential equations, it is possible to view all of the solution function graphs in terms of "prototypical" graphs. A prototypical solution graph represents an infinite number of other solution graphs. For example, in part (i) below one can view the entire family of functions that solve the differential equation in terms of two different prototypical solution graphs separated by an equilibrium solution: one prototypical solution graph is above the t-axis and one is below the t-axis. Each is prototypical because it can stand for all other solution graphs (in its respective region) through horizontal translation. Recall the "Making Connections" section of Unit 3.

   (i) $\dfrac{dy}{dt} = -y$ (ii) $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{2}\right)$ (iii) $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{2}\right) + 3$ (iv) $\dfrac{dy}{dt} = y^2$

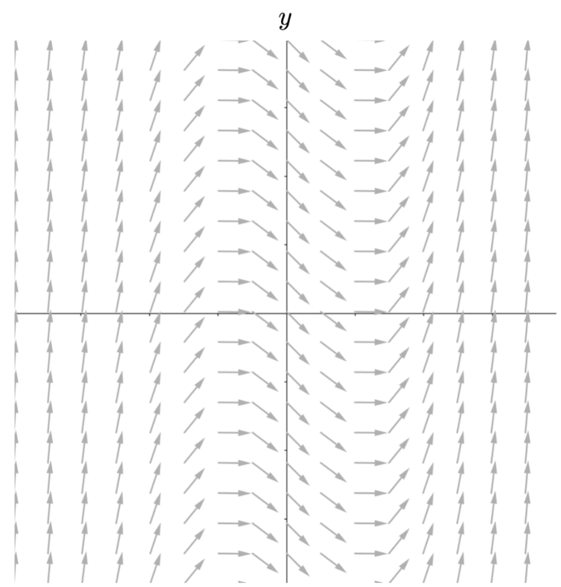
   (a) For each differential equation above, draw a phase line and representative graphs of solutions.

   (b) For each differential equation above, explain how your response to number 1a can be interpreted in terms of prototypical solutions separated by equilibrium solutions.

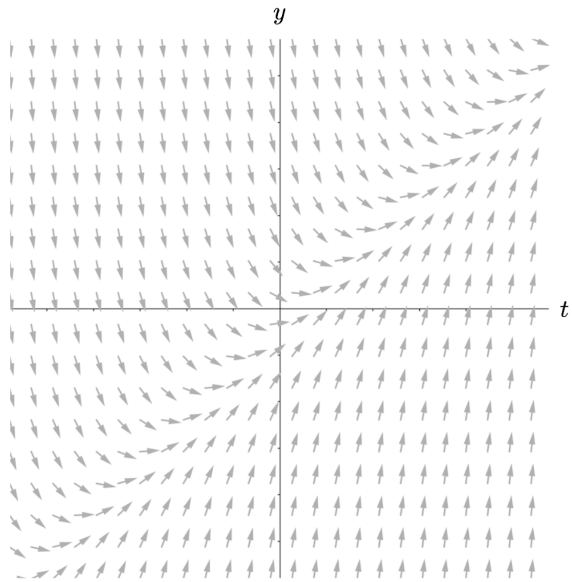
2. For each of the following slope fields, create a differential equation whose slope field would be similar to the one given. Give reasons for why you created the differential equation as you did. You may create whatever scale on the axes that you want.



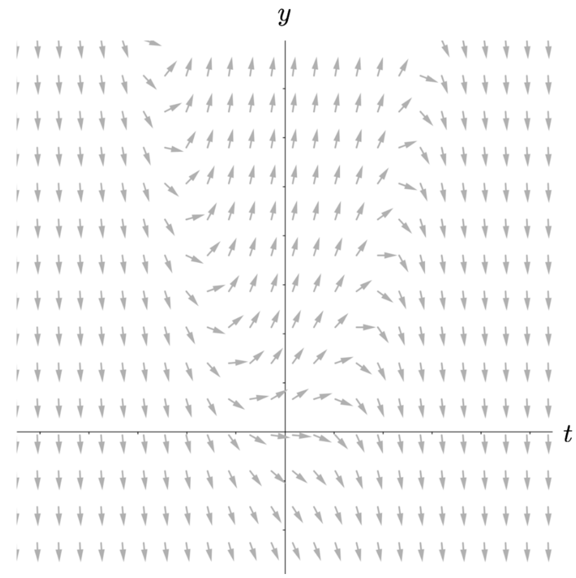
(a)



(b)



(c)



(d)

3. For each part below, create a continuous, autonomous differential equation that has the stated properties (if possible). Explain how you created each differential equation and include all graphs or diagrams you used and how you used them. If it is not possible to come up with a differential equation with the stated properties, provide a justification for why it cannot be done.

   (a) Exactly three constant solution functions, two repellers and one attractor.

   (b) Exactly two constant solution functions, one a repeller and one a node.

   (c) Exactly two constant solution functions, both attractors.

4. For each part below, create an autonomous differential equation that satisfies the stated criteria

   (a) $y(t) = 0$ and $y(t) = -4$ are the only constant solution functions

   (b) $y(t) = e^{-t+1}$ is a solution

   (c) $y(t) = e^{2t-5}$ is a solution

   (d) $y(t) = 10e^{0.3t}$ is a solution

   (e) $y(t) = 1 - e^{-t}$ and $y(t) = 1 + e^{-t}$ are solutions

5. For a phase line to be a meaningful tool, explain why it is essential for the differential equation to be autonomous.

6. In class you and your classmates continue to develop creative and effective ways of thinking about particular ideas or problems. Discuss at least one idea or way of thinking about a particular problem that has been discussed in class (either in whole class discussion or in small group) that was particularly helpful for enlarging your own thinking and/or that you disagreed with and had a different way of thinking about the idea or problem.

Cooling Coffee

A group of students want to develop a rate of change equation to describe the cooling rate for hot coffee in order that they can make predictions about other cups of cooling coffee. Their idea is to use a temperature probe to collect data on the temperature of the coffee as it changes over time and then to use this data to develop a rate of change equation.

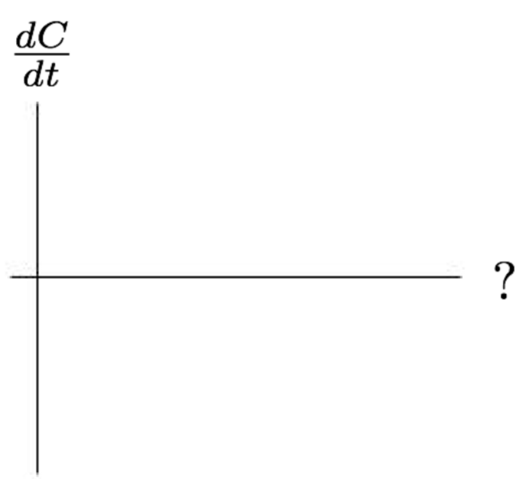
The data they collected is shown in the table below. The temperature C (in degrees Fahrenheit) was recorded every 2 minutes over a 14 minute period.

Time (min)	Temp. (°F)
0	160.3
2	120.4
4	98.1
6	84.8
8	78.5
10	74.4
12	72.1
14	71.5

1. Figure out a way to use this data to create a third column whose values approximate $\frac{dC}{dt}$, where C is the temperature of the coffee.

2. Do you expect $\frac{dC}{dt}$ to depend on just the temperature C, on just the time t, or both the temperature C and the time t?

3. Sketch below your best guess for the graph of $\frac{dC}{dt}$.



4. (a) Input the data from your extended table in question 1 into the GeoGebra applet https://ggbm.at/uj2gbz3V to plot points for $\frac{dC}{dt}$ vs. C. Does this plot confirm or reject your sketch from question 3?

(b) Toggle on the curve fitting tool and find an equation that fits your data.

5. One group of students figured out that a reasonable rate of change equation to be

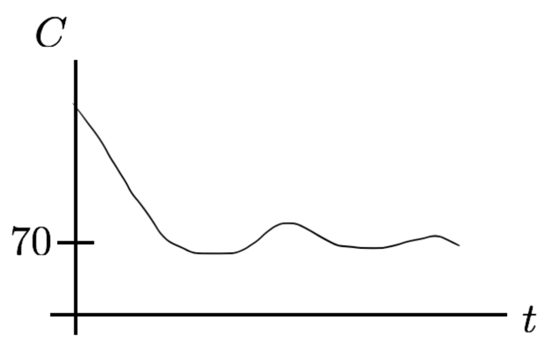
$$\frac{dC}{dt} = -0.4C + 28$$

which they rewrote as

$$\frac{dC}{dt} = -0.4\left(C - 70\right).$$

Interpret the meaning of the number 70 in this equation. Does this rate of change equation also make sense for predicting the future temperature of a glass of ice tea? Why or why not?

6. According to the rate of change equation from questions 4 and 5, is it possible for a graph of the **exact** solution to look like the one below? Why or why not? Give more than one reason for your answer.



Population Growth - Limited Resources

A group of biologists want to study the population growth of certain bacteria in a laboratory. The scientists realized that the culture for the bacteria does not provide unlimited resources. Hence, the rate of change equation $\dfrac{dP}{dt} = kP$ is not appropriate. They conducted experiments to determine how the rate of change of population depends on just the population. The data they collected is shown in the table below (numbers are properly scaled). At various population levels, the scientists measured the population after one day.

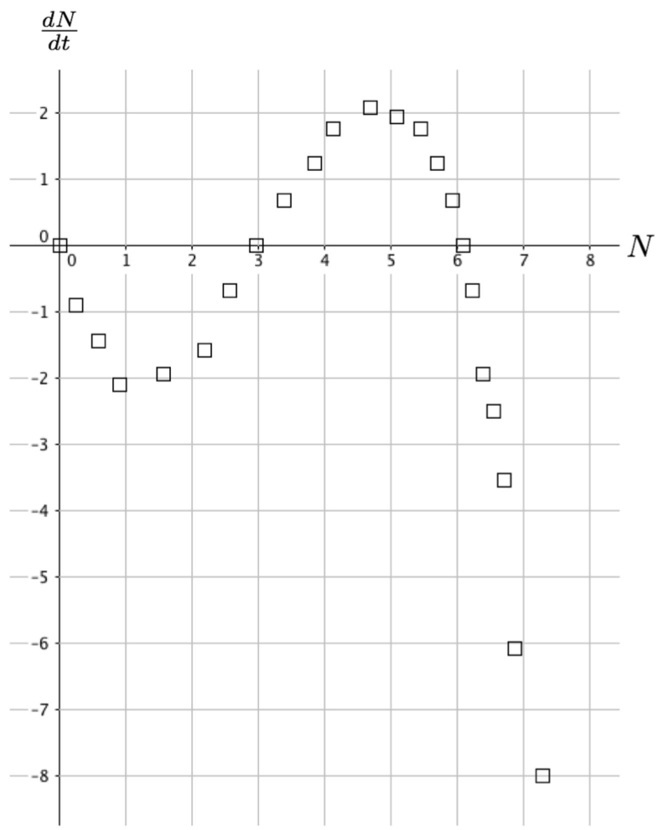
Beginning Population	Population after one day
2	2.34
4	4.54
6	6.62
8	8.58
10	10.40
12	12.10
14	13.66
16	15.10
18	16.42
20	17.60

7. Create a third column whose values approximate $\frac{dP}{dt}$. Explain why the method you used to create this column makes sense.

8. In this course we will call a graph of $\frac{dP}{dt}$ vs. P, when $\frac{dP}{dt}$ is an autonomous differential equation, an **Autonomous Derivative Graph**. Create an **autonomous derivative graph** and figure out a way to analyze this graph to determine the long term behavior for each of the beginning populations given in the table above.

Analyzing Graphs of Autonomous Differential Equations

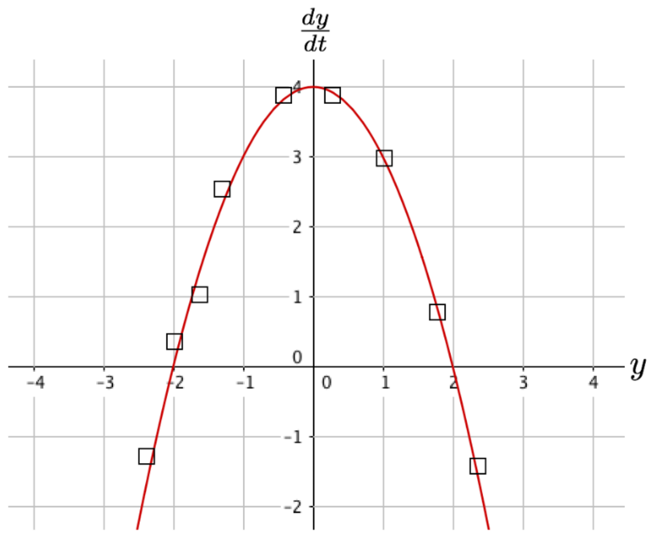
9. A group of biologists is studying a particular bug population in a rainforest. They gathered data about these bugs for different population values, N, at different times, t. The scientists reasoned that the rate of change depended only on the population and not on time. They approximated the derivatives $\frac{dN}{dt}$ (as was done with the cooling coffee from before) and plotted the autonomous derivative graph, as seen below:



For each part below, use the autonomous derivative graph to predict what the ultimate fate of the population will be. Describe (in words) the long-term behavior of each solution corresponding to the given initial condition. In addition, illustrate your conclusions with a suitable graph or graphs and classify all equilibrium solutions as either an attractor, repeller, or node.

(a) $N(0) = 2$
(b) $N(0) = 3$
(c) $N(0) = 4$
(d) $N(0) = 4.5$
(e) $N(0) = 6$
(f) $N(0) = 8$

10. Below is an autonomous derivative graph. Figure out the long-term behavior of every possible solution function and illustrate your conclusions with a suitable graph or graphs.



Homework Set 7

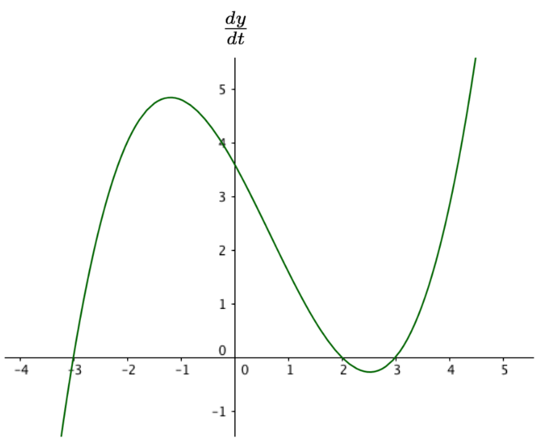
1. For this problem, use the coffee cooling rate of change equation

$$\frac{dC}{dt} = -0.4C + 28.$$

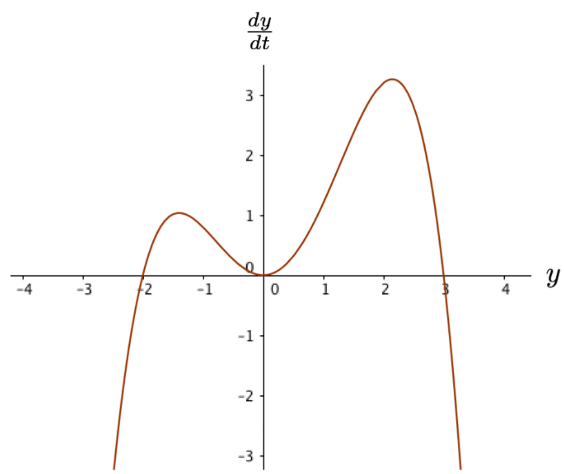
   (a) Is there ever a time when two cups of coffee, one at initially 160°F and one at 180°F, are the exact same temperature? Answer this question according to the uniqueness theorem. Comment on whether your answer matches what you expect to happen in real life?

   (b) How long will it take a cup of hot coffee that is initially 180°F to cool down to 100°F? Use the reverse product rule to figure this out and then check the reasonableness of your answer with Euler's method.

2. For each part below you are provided with an autonomous derivative graph. Figure out the long-term behavior of every possible solution function. Illustrate your conclusions with representative $y(t)$ solution graphs and summarize your findings about the long-term behavior of different solutions in paragraph form.



(a)



(b)

3. For each part in problem 2, create a phase line and classify each equilibrium solution as either an attractor, repeller, or node.

4. For problem 2b, use the uniqueness theorem to determine if any of the non-constant solution functions ever reach the equilibrium solution of $y(t) = 0$ in a finite amount of time.

5. Given an autonomous differential equation $\frac{dy}{dt} = f(y)$, give a general strategy for how to use an autonomous derivative graph to determine the long term behavior of solution functions.

6. Suppose you wish to predict future values of some quantity, y, using an autonomous differential equation (that is, dy/dt depends explicitly only on y). Experiments have been performed that give the following information:

   - The only equilibrium solutions are $y(t) = 0$, $y(t) = 15$, and $y(t) = 60$
   - If the value of y is 100, the quantity decreases
   - If the value of y is 30, the quantity increases
   - If the value of y is negative, the quantity increases

   (a) How many different phase lines match the above? Sketch all possible phase lines.

   (b) Provide a rough sketch of an autonomous derivative graph for each of your phase lines in part 6a.

   (c) For each of your different sketches in part 6b, develop a differential equation that fits the basic features.

7. In what ways is the letter y in the differential equation $\frac{dy}{dt} = .3y$ both a variable and a function? In what ways is $\frac{dy}{dt}$ a function?

8. Newton's law of cooling is an empirical law that states that an object immersed in a constant, ambient temperature will have its temperature change at a rate proportional to the difference between the its temperature and the ambient temperature. Explain how the cooling coffee problem reflects Newton's law of cooling.

9. A body was found in a temperature controlled environment (i.e., you know the room temperature) and is subject to Newton's law of cooling. Explain why you only need the room temperature and the measurement of the body's temperature at two different times to give an estimate of the time of death.

Fish Harvesting

A mathematician at a fish hatchery has been using the differential equation $\frac{dP}{dt} = 2P\left(1 - \frac{P}{25}\right)$ as a model for predicting the number of fish that a hatchery can expect to find in their pond.

1. Use an autonomous derivative graph, a phase line, and a slope field to analyze what this differential equation predicts for future fish populations for a range of initial conditions. Present all three of these representations and describe in a few sentences how to interpret them.

2. Recently, the hatchery was bought out by fish.net and the new owners are planning to allow the public to catch fish at the hatchery (for a fee of course). This means that the previous differential equation used to predict future fish populations needs to be modified to reflect this new plan. For the sake of simplicity, assume that this new plan can be taken into consideration by including a constant, annual harvesting rate k into the previous differential equation. Below are two modifications to the differential equation that may account for the new plan, as well as an option to create your own modification. Do you agree with (a) or (b)? If yes, explain why. If no, create your own modification and explain your reasoning.

(a) $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P}{25}\right) - kP$

(b) $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P-k}{25}\right)$

(c) Create Your Own

3. Your team of consultants settled on $\dfrac{dP}{dt} = 2P\left(1 - \dfrac{P}{25}\right) - k$ to model the new fishing plan. Analyze the effect of different choices for the value of k on the fish population. Synthesize your analysis in a **one page** report for the new owners that illustrates the implications that various choices of k will have on future fish populations. Your report may include one or more graphical representations but must communicate the effect of different k values in a concise way.

4. In studying climate, scientists are often concerned about positive feedback loops: two or more processes that amplify each other, creating a system of amplification that leads to a vicious cycle. One example is the interaction of water vapor with global temperature. If global temperature increases, the capacity of the atmosphere to contain evaporated water vapor also increases. If water resources are available, this would result in an increased amount of water vapor in the atmosphere. Water vapor is a greenhouse gas, thus if a climate system has more water vapor in the atmosphere, the global temperature will increase due to the increased insulation of the atmosphere. This positive feedback loop will eventually equilibrate at a higher temperature. Some scientists predict that a global increase in average temperature of just two degrees would be enough to kick off a system of positive feedback loops that would equilibrate at a temperature at least 6 degrees higher than we have now. This 6-degree increase would be enough to turn rainforests into deserts and melt ice caps. It may even redistribute the areas of the world that can support human life, i.e. making previously uninhabitable places, like the northern reaches of Siberia and Canada, inhabitable (though they may not support agriculture) and previously inhabitable places, like coastal cities, uninhabitable.

   (a) A modern pre-industrial average temperature at the equator is about 20 degrees Celsius. Assuming that our current global climate system has not undergone this vicious cycle, model this system with a phase line. What are the essential features of that phase line?

   (b) What is a simple differential equation that corresponds to your above phase line?

(c) A group of scientists came up with the following model for this global climate system:

$$\frac{dC}{dt} = \frac{1}{10}\Big(C - 20\Big)\Big(22 - C\Big)\Big(C - 26\Big) - k$$

where C is the temperature, in Celsius, and k is a parameter that represents governmental regulation of greenhouse gas emissions. Assume the baseline regulation corresponds to $k = 0$, increasing regulation corresponds to increasing k, and the current equatorial temperature is around 20 degrees. To what equatorial temperature will the global climate equilibrate?

(d) Sketch a bifurcation diagram and use it to describe what happens to the global temperature for various values of k.

(e) Suppose at the start of a new governmental administration, the temperature at the equator is about 20 degrees Celsius, and $k = 0$. Based on the model and other economic concerns, a government decides to deregulate emissions so that $k = -0.5$. Later, the Smokestack Association successfully lobbied for a 5% change, resulting in $k = -0.525$. Subsequently, a new administration undid that change, reverting to $k = -0.5$, and eventually back to $k = 0$. What is the equilibrium temperature at the equator after all of these changes?

(f) Use your bifurcation diagram to propose a plan that will return the temperature at the equator to 20 degrees Celsius.

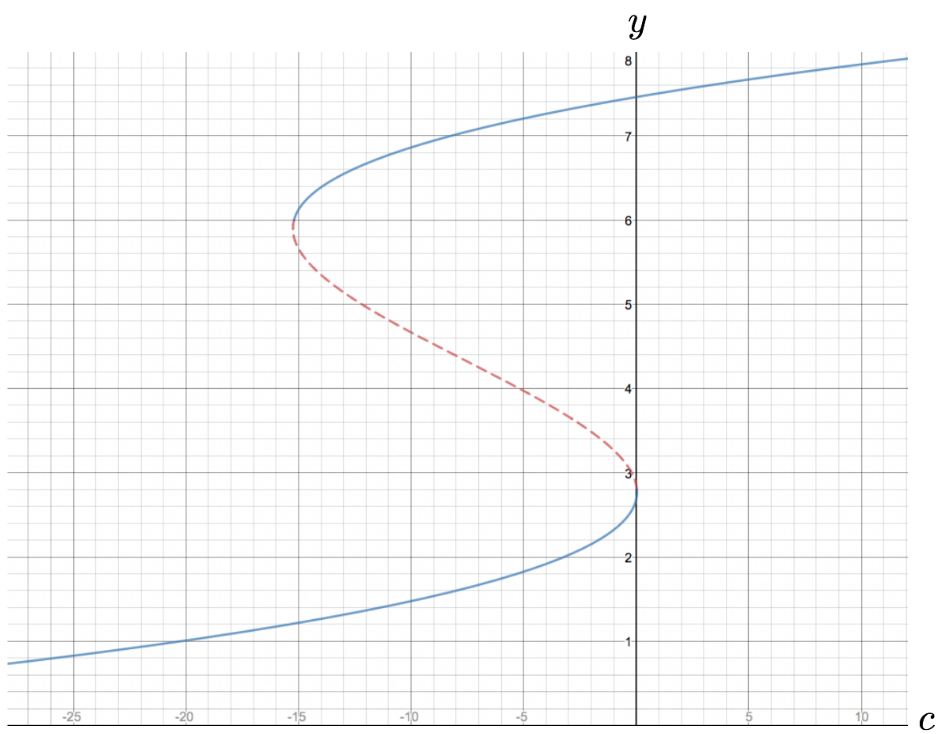
Homework Set 8

1. (a) The owners of fish.net have settled on model $dP/dt = 2P(1 - P/25) - k$ to make their business decisions, where P is the number of fish in thousands, and k is a harvesting rate measured in thousands of fish per year. They initially allow a harvesting rate $k = 12$. If they allow fishing to continue for a while at this rate, what does their model predict for the long term number of fish in the lake?

   (b) The early years of fish harvesting went well, so they increased the harvesting rate by a modest amount. They now allow harvesting rate corresponding to $k = 13$. What does this model predict will be the long term result of this fishing practice?

   (c) The owners of fish.net panicked when their fish population reached $P = 5$ and decided to return to their original business model with $k = 12$. Will the fish population return to the levels you described in problem 1a? Why or why not?

2. The bifurcation diagram for an autonomous differential equation $dy/dt = f(y)$ is shown below. The solid parts corresponds to stable equilibria and the dashed part is for unstable ones. $f(y)$ has a parameter c, and changing the value of that parameter changes the behavior of the system, as shown.



(a) Sketch the phase lines when $c = -20$, $c = -5$, $c = 0$, and $c = 10$.

(b) Sketch the corresponding graphs of y vs. t for each of the choices of c listed above.

(c) For what values of c does the system have two attractors?

(d) As shown, the bifurcation diagram has two stable (solid) "branches" connected by an unstable (dashed) branch. Would it be possible for the entire curve to be stable? Why or why not?

(e) If this model represents a physical system, and you measure that the system has a steady state of $y = 2$, what value of c should you choose for your model?

(f) Again, let's think of this model as representing some physical system, similar to the hatchery example we considered in class. You are the owner of that system, and you have control over the value of c. $y(t)$ represents the state of your system at a given time. Consider the following experiment.

i. Let's say the system starts with an initial condition of $y(0) = 0$, and you fixed c at $c = -10$. After a long time elapses, what value does y approach?

ii. Assume that y has evolved to your answer in problem 2(f)i, and that result is not something you are completely happy with. You've heard that a company down the road is using $c = 10$, so you make that change. What value does y approach now (after substantial time has passed)?

iii. Assume that y has evolved now to your answer in 2(f)ii. Unfortunately, this new value of y is even worse than the old one, so you want to change c back to $c = -10$. Will the system evolve back to your answer in problem 2(f)i? Explain.

3. For each of the following, illustrate with suitable solution function graphs and/or phase lines the way in which the solutions change as the value of r changes. Identify the precise value(s) of r for which there is a either a change in the number of equilibrium solution(s) or a change in the type of equilibrium solution(s). Explain in words the change that happens at each significant value of r identified.

(a) $\dfrac{dy}{dt} = (y-3)^2 + r$

(b) $\dfrac{dy}{dt} = y^2 - ry + 1$

(c) $\dfrac{dy}{dt} = ry + y^3$

(d) $\dfrac{dy}{dt} = y^6 - 2y^4 + r$

4. For problem 3a, sketch a graph of the equilibrium solutions as r varies. Such a graph is referred to as "bifurcation diagram" and the significant values of r are called "bifurcation values."

5. For problem 3b, sketch a bifurcation diagram and identify the bifurcation values.

6. For problem 3c, sketch a bifurcation diagram and identify the bifurcation values. Why might this bifurcation be called a "pitchfork bifurcation?"

Rabbits and Foxes

Most species live in interaction with other species. For example, perhaps one species preys on another species, like foxes and rabbits. Below is a **system of rate of change equations** intended to predict future populations of rabbits and foxes over time, where R is the population (in hundreds or thousands, for example) of rabbits at any time t and F is the population of foxes at any time t (in years).

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

1. (a) In earlier work with the rate of change equation $\frac{dP}{dt} = kP$ we assumed that there was only one species, that the resources were unlimited, and that the species reproduced continuously. Which, if any, of these assumptions is modified and how is this modification reflected in the above system of differential equations?

   (b) Interpret the meaning of *each* term in the rate of change equations (e.g., how do you interpret or make sense of the $-1.4RF$ term) and what are the implications of this term on the future predicted populations? Similarly for $3R$, $-F$, and $0.8RF$.

2. (a) Scientists studying a rabbit-fox population estimate that the current number of rabbits is 1 (scaled appropriately) and that the scaled number of foxes is 1. Use two steps of Euler's method with step size of 0.5 to get numerical estimates for the future number of rabbits and foxes as predicted by the differential equations.

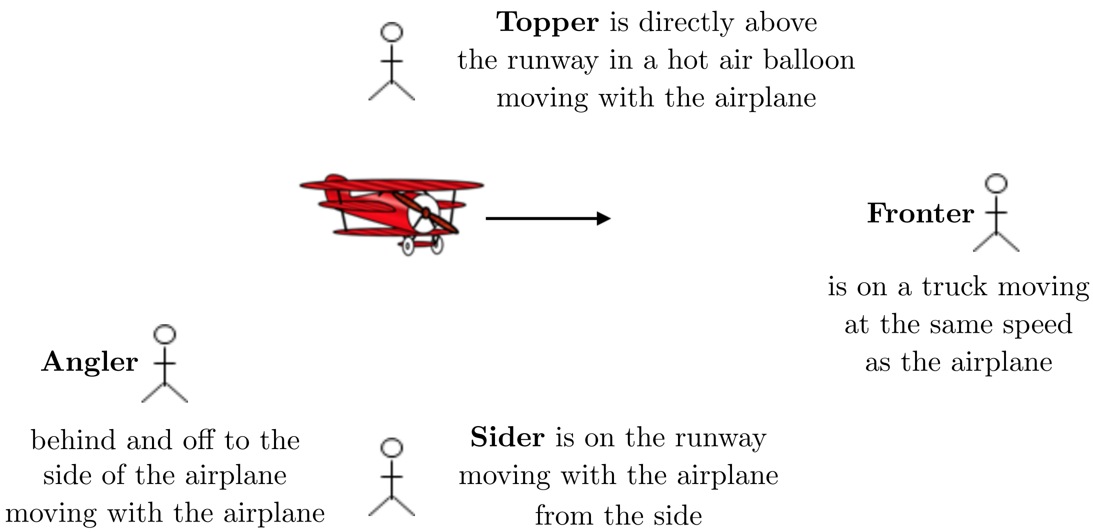
t	**R**	**F**
0	1	1
0.5		
1.0		

   (b) What are some different two dimensional and three dimensional ways to graphically depict your (t, R, F) values?

Three Dimensional Visualization

3.  A crop duster plane with a two blade propeller is rolling along a runway. On the end of one of the propeller blades, which are rotating clockwise at a slow constant speed, is a noticeable red paint mark. Imagine that for the first several rotations of the propeller blades the red mark leaves a "trace" in the air as the plane makes its way down the runway.

   (a) Simulate this scenario over time with a pipe cleaner. On appropriate combinations of the x, y, and t axes, sketch what Angler, Sider, Fronter, and Topper would ideally see assuming that they could always see the red mark. What view do you think is best and why?



Sketch your ideas for each of the following:

   (b) What if there was another paint mark on other end of the propeller, what, **ideally**, do the four observers see then? How does the trace of this mark relate to the previous trace?

   (c) What if there was a paint mark on the center of the propeller blade mechanism. What do the observers ideally see then?

   (d) How ideally would each observer see all of the above paint marks simultaneously?

4. (a) For the system of differential equations from problem 1,

$$\begin{aligned}\frac{dR}{dt} &= 3R - 1.4RF \\ \frac{dF}{dt} &= -F + 0.8RF\end{aligned}$$

consider the perspectives of Angler, Sider, Fronter, and Topper. What are the coordinate axes that correspond to each?

(b) Use the GeoGebra applet https://ggbm.at/U3U6MsyA to generate predictions for the future number of rabbits and foxes if at time 0 we initially have 3 rabbits and 2 foxes (scaled appropriately). Generate and reproduce below the perspectives of Angler, Sider, Fronter, and Topper from the crop duster problem.

(c) Use the same GeoGebra applet from problem 4b to experiment with different initial conditions and interpret the nature of the numerical solutions in the context of Rabbits and Foxes.

(d) Determine an initial rabbit and fox population at time 0 such that the 3D graph of the solution (Angler's view) is a shift of the 3D graph in problem 4b along the t-axis. What connections does this problem have to do with your study of autonomous first order differential equations?

5. (a) Suppose the current number of rabbits is 3 and the number of foxes is 0. Without using any technology and without making any calculations, what does the system of rate of change equations (same one as problem 4a) predict for the future number of rabbits and foxes? Explain your reasoning.

   (b) Use the same GeoGebra applet from problem 4b to generate the 3D plot and all three different views or projections of the 3D plot. Show each graph and explain how each illustrates your conclusion in problem 5a.

   (c) Using Fronter's view with initial condition $R = 3$ and $F = 2$, tell the story of what happens to the rabbit and fox population as time continues.

6. (a) What would it mean for the rabbit-fox system to be in equilibrium? Are there any equilibrium solutions to this system of rate of change equations? If so, determine all equilibrium solutions and generate the 3D and other views for each equilibrium solution.

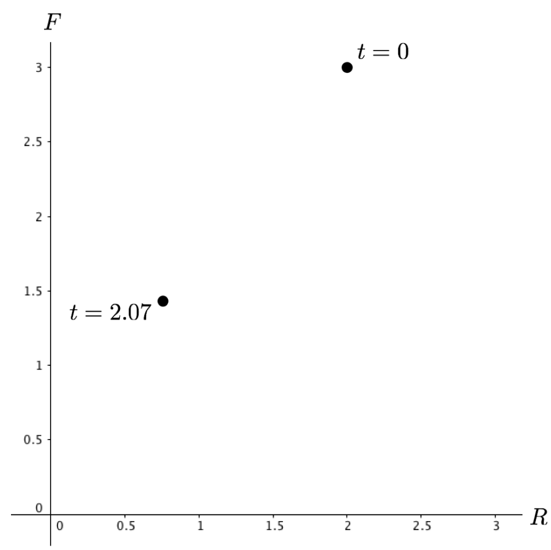
   (b) For single differential equations, we classified equilibrium solutions as attractors, repellers, and nodes. For each of the equilibrium solutions in the previous problem, create your own terms to classify the equilibrium solutions in 6a and briefly explain your reasons behind your choice of terms.

7. A group of scientists wants to graphically display the predictions for many different non-negative initial conditions (this includes 0 values for R and F, but not negative values) to the rabbit-fox system of differential equations and they want to do so using only one set of axes. What one single set of axes would you recommend that they use ($R - F - t$ axes, $t - R$ axes, $t - F$ axes, or $R - F$ axes)? Explain.

8. One view of solutions for studying solutions to systems of autonomous differential equations is the $x-y$ plane, called the **phase plane**. The phase plane, which is Fronter's view from the crop duster problem, is the analog to the phase line for a single autonomous differential equation.

   (a) Consider the rabbit-fox system of differential equations and a solution graph, as viewed in the phase plane (that is, the R-F plane), and the two points in the table below. These two points are on the same solution curve. Recall that the solutions we've seen in the past are closed curves, but notice that the solution could be moving clockwise / counterclockwise. Fill in the following table and decide which way the solution should be moving, and explain your reasoning.

t	R	F	dR/dt	dF/dt	dF/dR
0	2	3			
2.07	0.756	1.431			



$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

   (b) On the same set of axes from problem 8a plot additional vectors at the following points and state what is unique about these vectors.

R	F	dR/dt	dF/dt	dF/dR
1.25	0			
1.25	1			
1.25	2			
1.25	3			

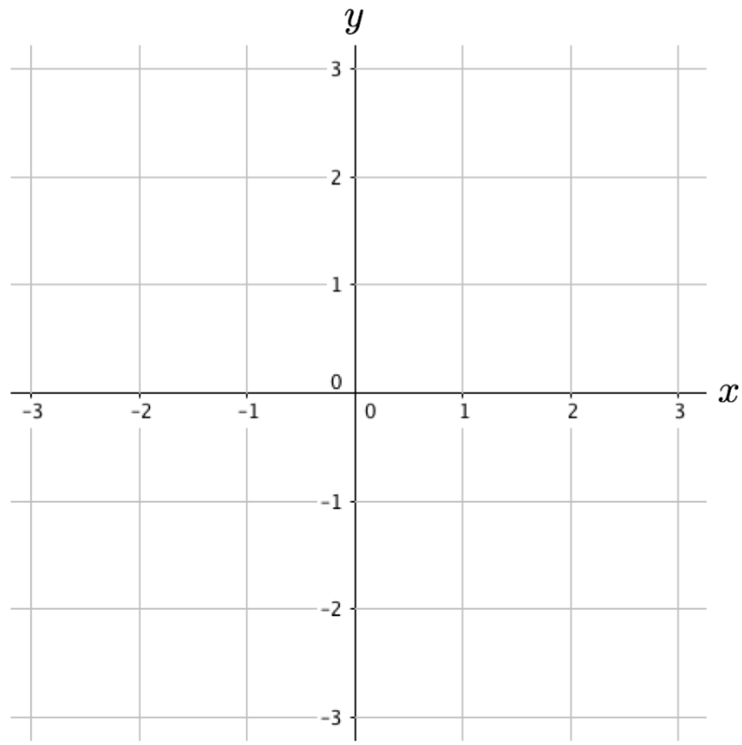
Vector Fields

Slope fields are a convenient way to visualize solutions to a single differential equation. For systems of autonomous differential equations the equivalent representation is a **vector field**. Similar to a slope field, a vector field shows a selection of vectors with the correct slope but with a normalized length. In the previous problem you plotted a few such vectors but typically more vectors are needed to be able to visualize the solution in the phase plane.

9. On a grid where x and y both range from -3 to 3, plot by hand a vector field for the system of differential equations

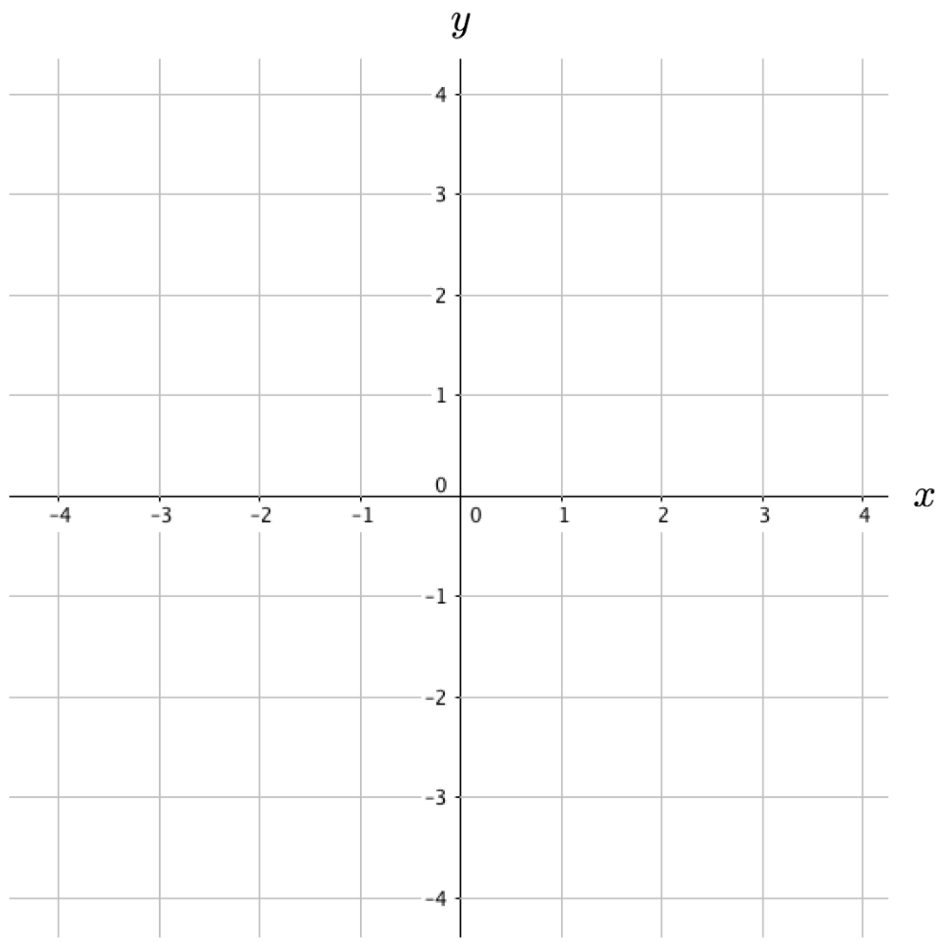
$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y$$

and sketch in several solution graphs in the phase plane.



10. (a) You may have noticed in problem 9 that along $x = 0$ all the vectors have the same slope. Similarly for vectors along the $y = x$. Any line or curve along which vectors all have the same slope is called an **isocline**. An isocline where $dx/dt = 0$ is called an **x-nullcline** because there is the horizontal component to the vector is zero and hence the vector points straight up or down. An isocline where $dy/dt = 0$ is called a **y-nullcline** because the vertical component of the vector is zero and hence the vector points left or right. On a grid from -4 to 4 for both axes, plot all nullclines for the following system:

$$\frac{dx}{dt} = 3x - 1.4xy$$
$$\frac{dy}{dt} = -y + 0.8xy$$



(b) How do these nullclines point to the cyclic nature of the Rabbit-Fox system?

11. A certain system of differential equations for the variables R and S describes the interaction of rabbits and sheep grazing in the same field. On the phase plane below, dashed lines show the R and S nullclines along with their corresponding vectors.

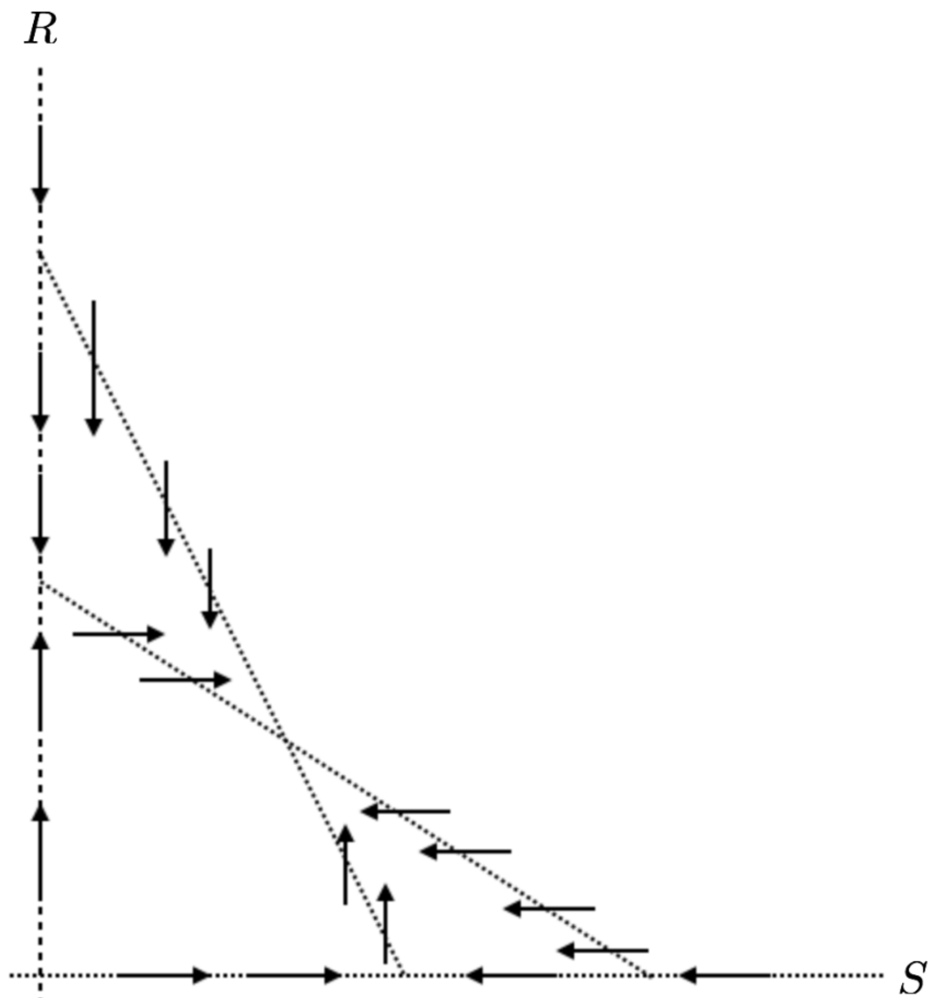
    (a) Identify the R nullclines and explain how you know.

    (b) Identify the S nullclines and explain how you know.

    (c) Identify all equilibrium points.

    (d) Notice that the nullclines carve out 4 different regions of the first quadrant of the RS plane. In each of these 4 regions, add a prototypical-vector that represents the vectors in that region. That is, if you think the both R and S are increasing in a certain region then, draw a vector pointing up and to the right for that region.

    (e) What does this system seem to predict will happen to the rabbits and sheep in this field?



Homework Set 9

1. (a) Consider again the crop duster plane problem but this time the red mark slowly drifts toward the center as the propellers rotate as the plane rolls along the runway. Sketch what the four observers see this time.

   (b) What do the four observers ideally see if the propellers are not rotating and the red mark drifts toward the center at a rate proportional to its distance from the center as the plane rolls along the runway?

2. Consider the same system of differential equations from problem 1. Use the GeoGebra applet https://ggbm.at/U3U6MsyA to generate predictions for the future number of rabbits and foxes if at time 0 we initially have the following different initial conditions: (i) 2 rabbits and 3 foxes, (ii) 1.5 rabbits and 4 foxes, and (iii) 4 rabbits and 2 foxes. For each of the different views, graph all three solutions on the same set of axes.

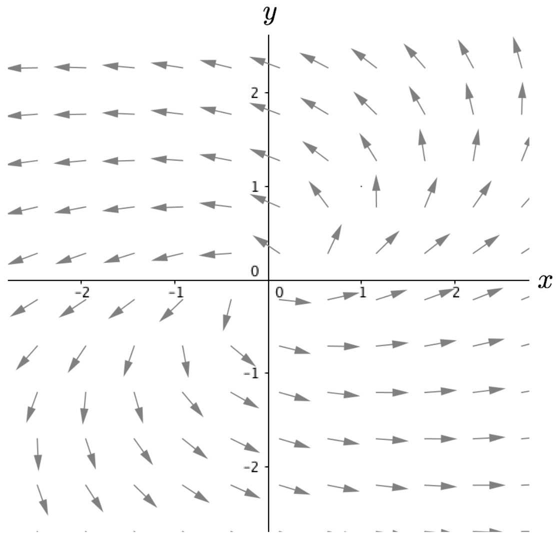
3. (a) Referring back to the rabbit and fox system of differential equations, suppose the current number of rabbits is 0 and the number of foxes is 2. Without using any technology and without making any calculations, what does the system of rate of change equations predict for the future number of rabbits and foxes? Explain your reasoning.

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

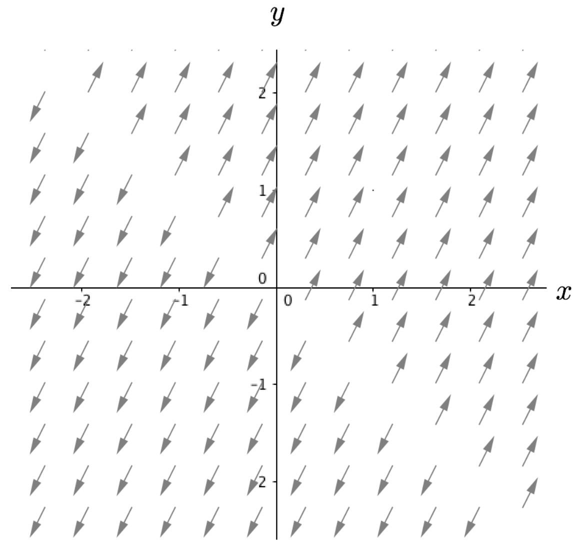
   (b) Use the GeoGebra applet to generate the 3D plot and all three different views or projections of the 3D plot. Show each graph and explain how each illustrates your conclusion in problem 3a).

   (c) Suppose the current number of rabbits is 0 and the number of foxes is 6. What does the system of rate of change equations predict for the future number of rabbits and foxes? How and why is this prediction related to the prediction when the initial number of rabbits is 0 and the number of foxes is 2?

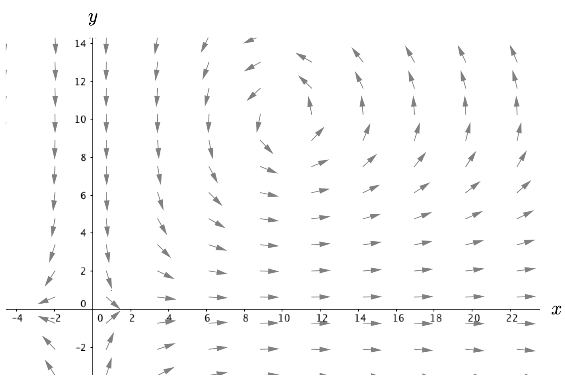
4. Here are three vector fields, A, B, and C. Below the vector fields are some pairs of rate of change equations. Determine which of the pairs match each of the vector fields. Write an explanation of each.



(a)



(b)



(c)

(i) $$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = -x + y$$

(ii) $$\frac{dx}{dt} = x - 0.1xy \qquad \frac{dy}{dt} = -y + 0.1xy$$

(iii) $$\frac{dx}{dt} = 2x - 3y \qquad \frac{dy}{dt} = x + y$$

(iv) $$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = 2x + 2y$$

5. In previous problems dealing with two species, one of the animals was the predator and the other was the prey. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

   (a) Which system of rate of change equations describes a situation where the two species compete and which system describes competitive species? Explain your reasoning.

$$\begin{array}{ll} \text{(A)} & \text{(B)} \\ \dfrac{dx}{dt} = -5x + 2xy & \dfrac{dx}{dt} = 3x(1-\dfrac{x}{3}) - \dfrac{1}{10}xy \\ \dfrac{dy}{dt} = -4y + 3xy & \dfrac{dy}{dt} = 2y(1-\dfrac{y}{10}) - \dfrac{1}{5}xy \end{array}$$

   (b) For system (A), plot all nullclines and use this plot to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

   (c) Use your results from 5b to sketch in the long-term behavior of solutions with initial conditions anywhere in the first quadrant of the phase plane. For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the first quadrant. Provide a sketch of your analysis in the x-y plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.

6. Consider the following systems of rate of change equations:

$$\begin{array}{ll} \underline{\textbf{System A}} & \underline{\textbf{System B}} \\ \dfrac{dx}{dt} = 3x(1-\dfrac{x}{10}) - \dfrac{1}{20}xy & \dfrac{dx}{dt} = 3x - \dfrac{xy}{100} \\ \dfrac{dy}{dt} = -5y + \dfrac{xy}{20} & \dfrac{dy}{dt} = 15y(1-\dfrac{y}{17}) + 25xy \end{array}$$

   In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

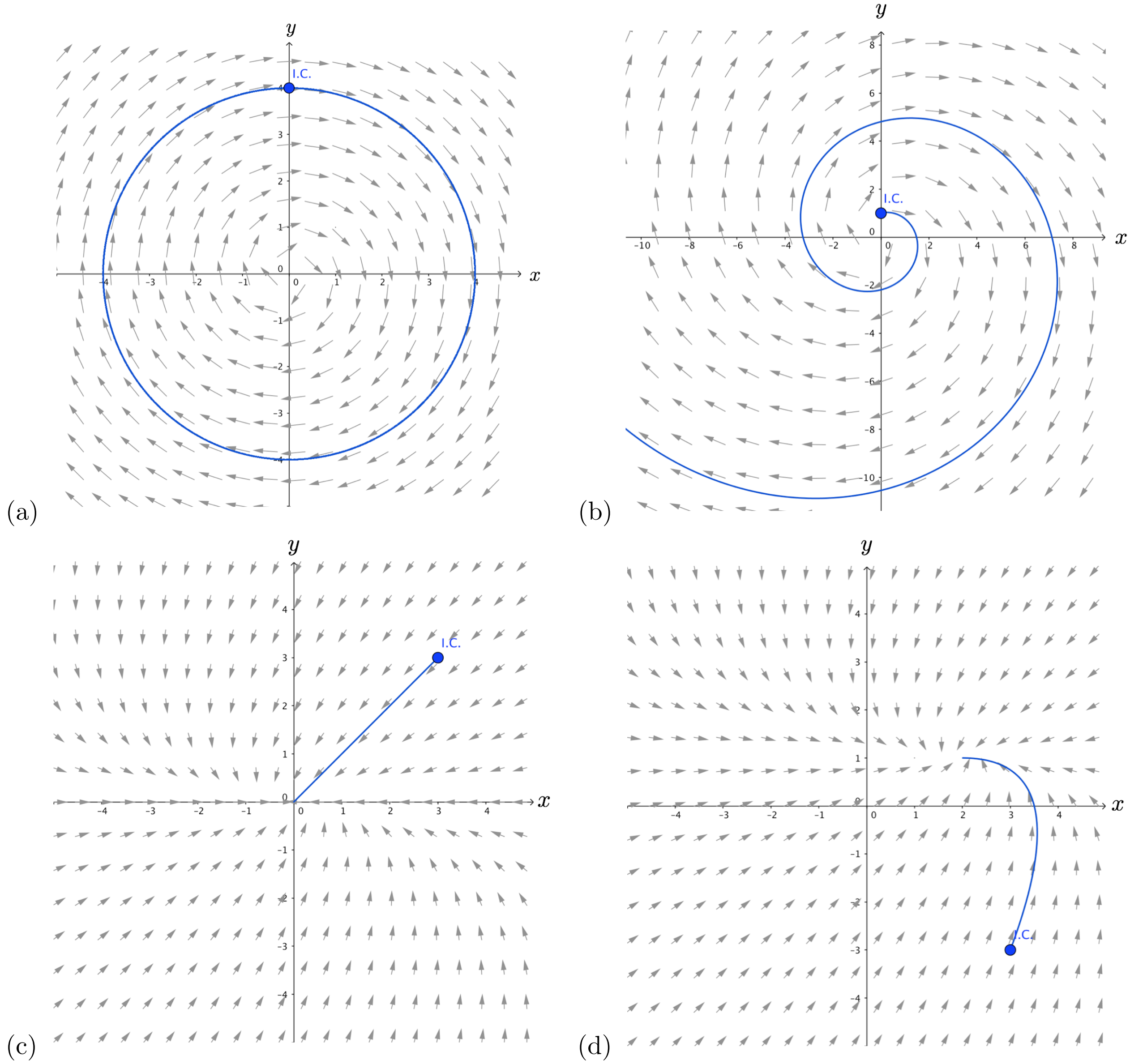
   (a) For both systems of differential equations, what does x represent? The predator or the prey? Explain.

   (b) What system represents predator and prey that are relatively the same size? Explain.

   (c) For system (A), plot all nullclines and use this plot to determine all equilibrium solutions. Verify your equilibrium solutions algebraically.

   (d) Use your results from 6c to sketch in the long-term behavior of solutions with initial conditions anywhere in the first quadrant of the phase plane. For example, describe the long-term behavior of solutions if the initial condition is in such and such region of the first quadrant. Provide a sketch of your analysis in the x-y plane and write a paragraph summarizing your conclusions and any conjectures that you have about the long-term outcome for the two populations depending on the initial conditions.

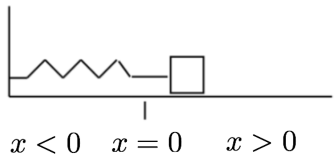
7. Provide sketches of x vs t and y vs t for each of the following phase planes and solution curves.



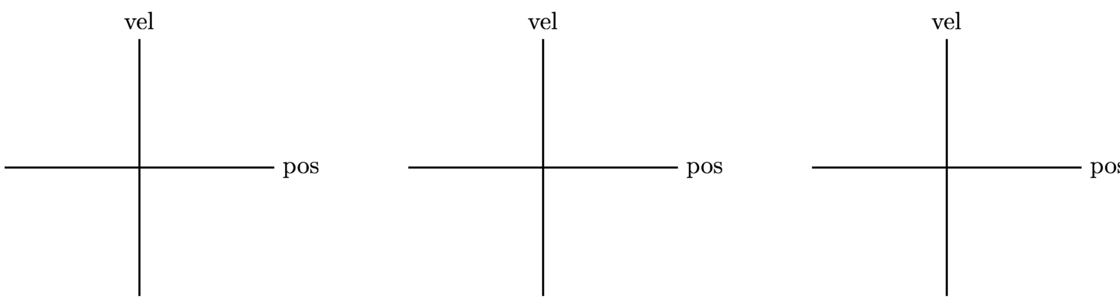
(a) (b) (c) (d)

Spring-Mass Motion Investigation

In this problem we use Newton's Law of motion ($\sum F = ma$) to develop a system of rate of change equations in order to be able to describe, explain, and predict the motion of a mass attached to a spring.



1. Depending on the values for parameters like the stiffness of the spring k, the weight of the object attached to the spring m, and the amount of friction, different behaviors may be possible. Imagine for a set spring and mass you vary the amount of friction on the surface. What do you imagine the various position versus velocity graphs would look like? Provide rough sketches.



2. Use Newton's Law of motion to develop a rate of change equation to model the motion of an object on a spring. Assume that the only forces acting on the object are the spring force ($-kx$, where k is the spring constant) and the friction force (assumed to be proportional to the velocity, namely $-b\frac{dx}{dt}$, where b is the damping coefficient).

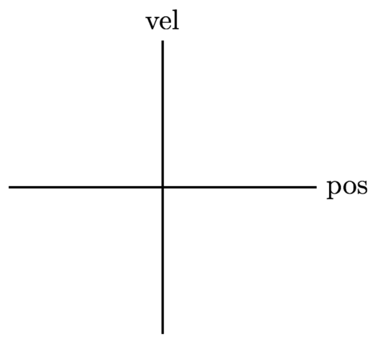
3. Application of Newton's Law of Motion to the spring-mass situation in the previous problem results in the following:

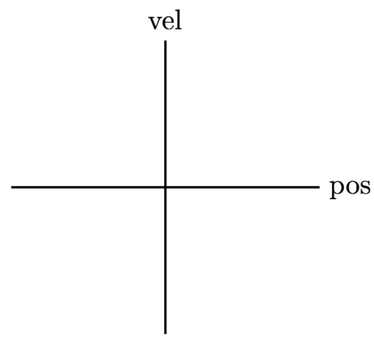
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0,$$

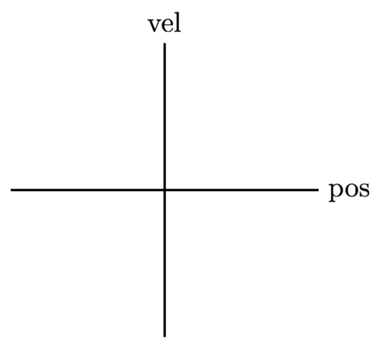
where x is the position of the object attached to the end of the spring, m is the mass of the object, b is the friction parameter (also called damping coefficient), and k is the spring constant. Because $\frac{dx}{dt} = y$, where y is the velocity, and $\frac{dy}{dt} = \frac{d^2x}{dt^2}$, we can converting this to a system of two differential equations as follows:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{k}{m}x - \frac{b}{m}y \end{aligned}$$

Use the GeoGebra applet https://ggbm.at/vT5tgWrg to investigate the motion of the object as depicted in the phase plane when $m = 1$, the spring constant $k = 2$, and the friction parameter, b, varies between 0 and 4. In particular, how does the vector field (and corresponding behavior of the mass) change when the friction parameter increases from 0 to say 2, 2.3, 3, or 3.8? Use the space below to record your observations.



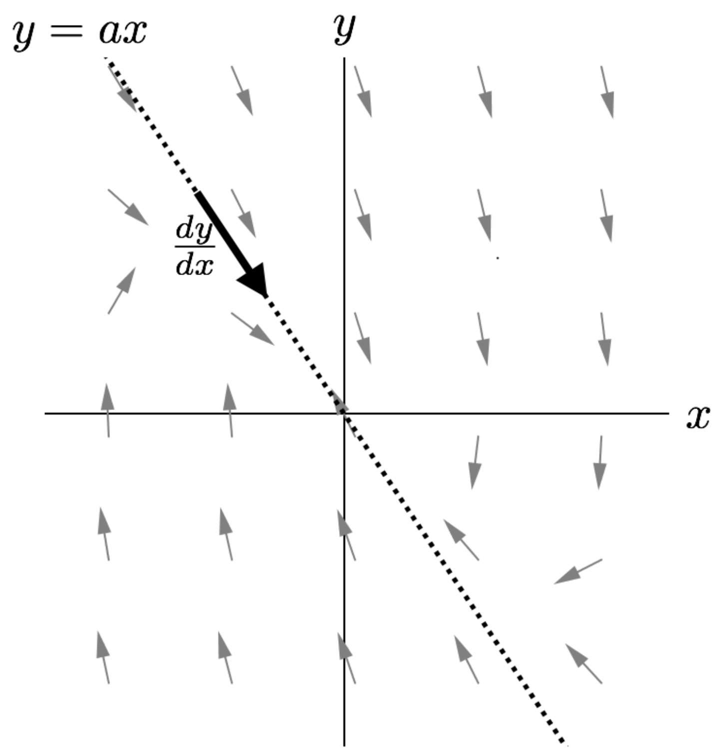




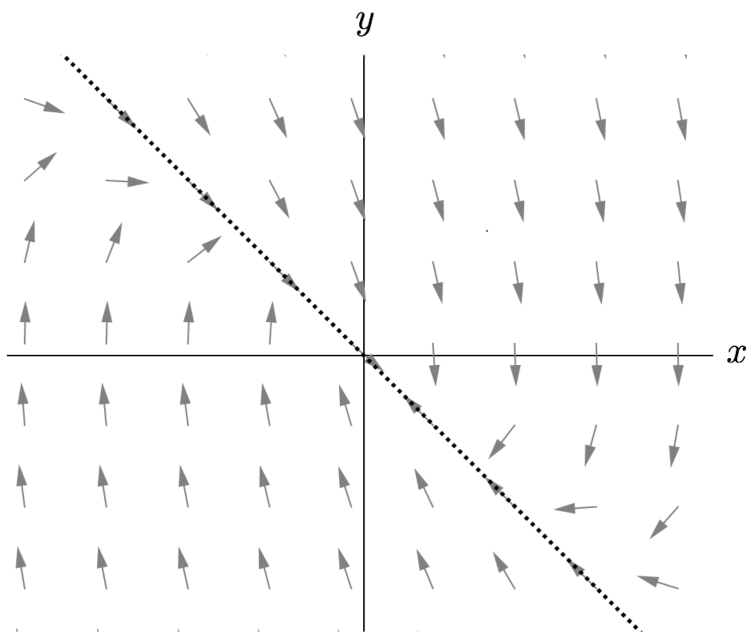
4. Joey and Kara set the friction parameter to 3, resulting in the following system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

They notice that graphs of solutions in the position-velocity plane seem to get pulled into the origin along a straight line. Help Joey and Kara figure out how to use algebra to find the slope of this straight line.



5. Continuing their investigation with the friction parameter set to 3, Kara and Joey are working to find the slope of the observed straight line. Joey sets up the equation $\dfrac{-2x - 3y}{y} = \dfrac{y}{x}$ and Kara sets up the equation $\dfrac{-2x - 3ax}{ax} = a$. Interpret Joey's and Kara's equations and then solve both.



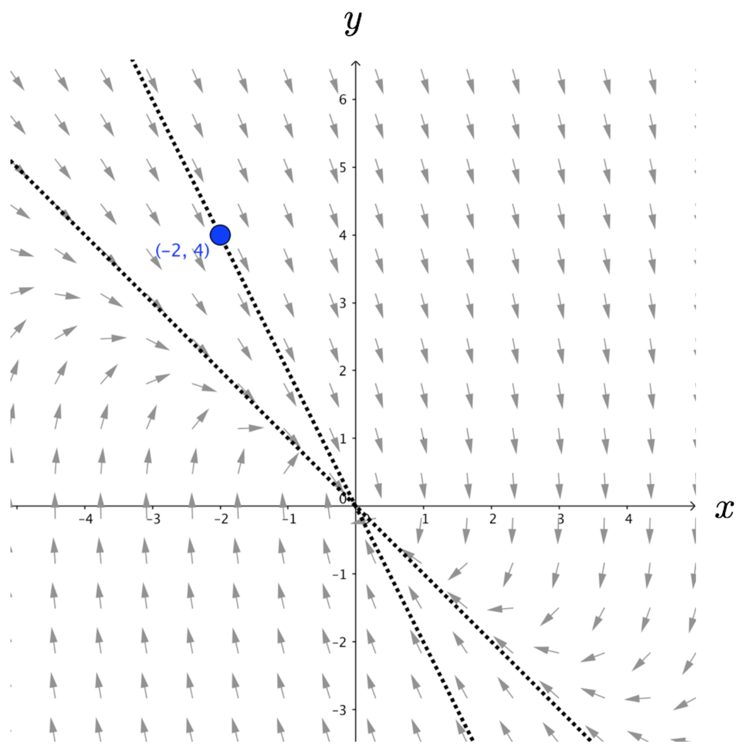
6. Place your finger on the dotted line starting in the second quadrant and trace out the path that the mass takes, as represented in the phase plane. Describe what happens to your finger and relate this to the motion of the mass.

7. Joey found another straight line solution when the friction parameter b was set to 1. Use algebra to find the slope or explain why he is mistaken.

8. In your investigation of the spring-mass system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

you should have found that when the friction parameter was equal to 3, solutions with initial conditions that are either on the line $y = -x$ or on the line $y = -2x$ head directly toward the origin along a straight path.



For the initial condition (-2, 4), what are the equations for $x(t)$ and $y(t)$? Hint: substitute $y = -2x$ and $x = -y/2$ into dx/dt and dy/dt, respectively.

9. Susan notices that the $x(t)$ and $y(t)$ equations have the same exponent, and then makes the conjecture that along any straight line solution, $x(t)$ and $y(t)$ **must** have the same exponent. Do you agree with her conjecture? Why or why not?

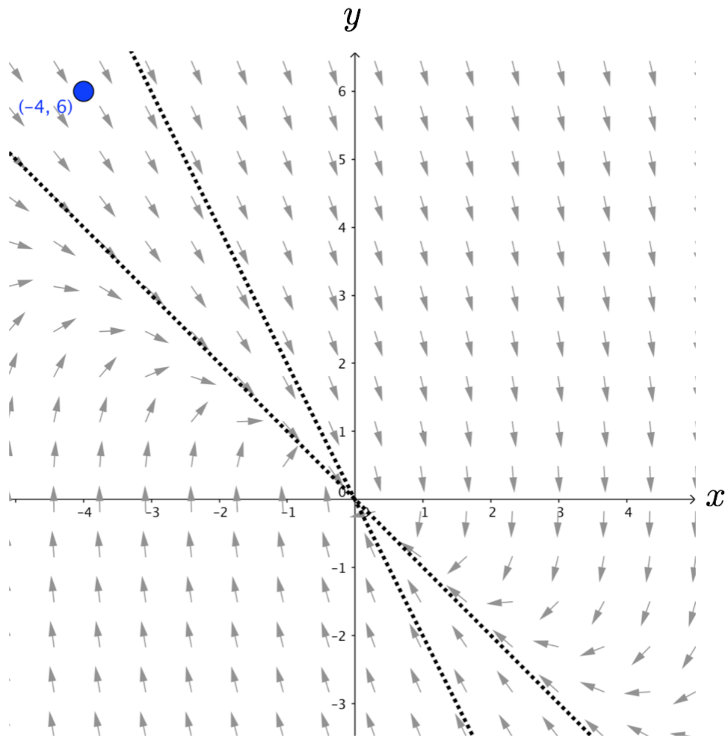
10. (a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition (-1, 2)? What does the 3D graph of this solution look like?

    (b) If you multiplied $x(t)$ and $y(t)$ equations from problem 10a by some number, say -3 for example, is the result also a solution to the system of differential equations? Algebraically show that your conclusion is correct.

    (c) What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -2x$?

11. For the initial condition (-2, 2), what are the equations for $x(t)$ and $y(t)$? What are the $x(t)$ and $y(t)$ equations for *any* solution with initial condition along the line $y = -x$?

12. (a) Suppose you were to start with an initial condition somewhere in the second quadrant between the two straight line solutions, say at (-4, 6). Sketch what you think the solution as viewed in the phase plane looks like and explain your reasoning.



(b) Notice that $(-4, 6)$ is a linear combination of the initial conditions $(-2, 4)$ and $(-2, 2)$, that is, $(-4, 6) = (-2, 4) + (-2, 2)$. Show that the solution with the initial condition (-4, 6) is also a linear combination of the solutions with initial conditions $(-2, 4)$ and $(-2, 2)$.

(c) According to your result in 12b, what does the solution in the phase plane look? Explain your reasoning.

13. (a) What are the $x(t)$ and $y(t)$ equations for the solution with initial condition $(2, 5)$?

(b) **According to your $x(t)$ and $y(t)$ equations**, what does the solution in the phase plane look like? Explain your reasoning and provide a sketch. Use the GeoGebra applet, https://ggbm.at/cMSUC7qR to corroborate your conclusion.

(c) Develop an argument that almost all graphs of solutions in the phase plane head into the origin with a slope of -1.

14. As a review of this unit, answer the following questions for the following system

$$\frac{dx}{dt} = -3x + 2y$$
$$\frac{dy}{dt} = 6x + y$$

(a) Find the slopes of the straight line solutions.

(b) For each straight line, find a solution.

(c) Form the general solution.

(d) In the phase plane sketch the straight line solutions and several non-straight line solutions.

(e) How would you classify the equilibrium point?

Homework Set 10

1. Consider the system from questions 3-7 from Unit 10. What is the smallest value of b for which we get solutions that, when viewed in the position-velocity plane, lie along a straight line? Algebraically support your conclusion.

2. Straight line Solutions for Systems of the Form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

Systems of equations of the form above are a special type of **linear system**. Linear systems model important applications, such as the spring mass system. Moreover, it is possible to find the general solution for any such linear system. For each of the system of differential equations below, address the following questions:

- How many equilibrium solutions are there are and what are they?
- Are there solutions that, when viewed in the phase plane (i.e., the $x - y$ plane), lie along a straight line? If so, algebraically figure out the exact slope of the straight line(s).
- For those systems that do have solutions that, when viewed in the phase plane, lie along a straight line, figure out the exact $x(t)$ and $y(t)$ equations for any solution with initial condition on the straight line(s).
- For those systems that have straight line solutions, write down the general solution.
- How would you classify the equilibrium solution? Create terms if needed to classify any new types of equilibrium solutions and explain the meaning of your terms.
- For those systems of differential equations that do have solutions that, when viewed in the phase plane, lie along straight lines, what do these straight lines look like in 3D? Provide your best 3D sketch.

(a) $\begin{aligned}\frac{dx}{dt} &= -3x + 2y \\ \frac{dy}{dt} &= 6x + y\end{aligned}$

(b) $\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -x + y\end{aligned}$

(c) $\begin{aligned}\frac{dx}{dt} &= -2x - 2y \\ \frac{dy}{dt} &= -x - 3y\end{aligned}$

(d) $\begin{aligned}\frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y\end{aligned}$

3. You figured out from our analysis on the previous problems, sometimes there are solutions in the phase plane that lie along a straight line headed directly towards or away from the equilibrium solution at the origin and sometimes there are not.

   (a) Explain in words how you figure out whether there are any straight line solutions in the phase plane and if so, what the slopes of this line or lines are. Demonstrate how your approach works in general for linear systems of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

   (b) Explain in words how you figure out the $x(t)$ and $y(t)$ equations for any and all straight line solutions in the phase plane. Demonstrate how your approach works in general for linear systems of the form

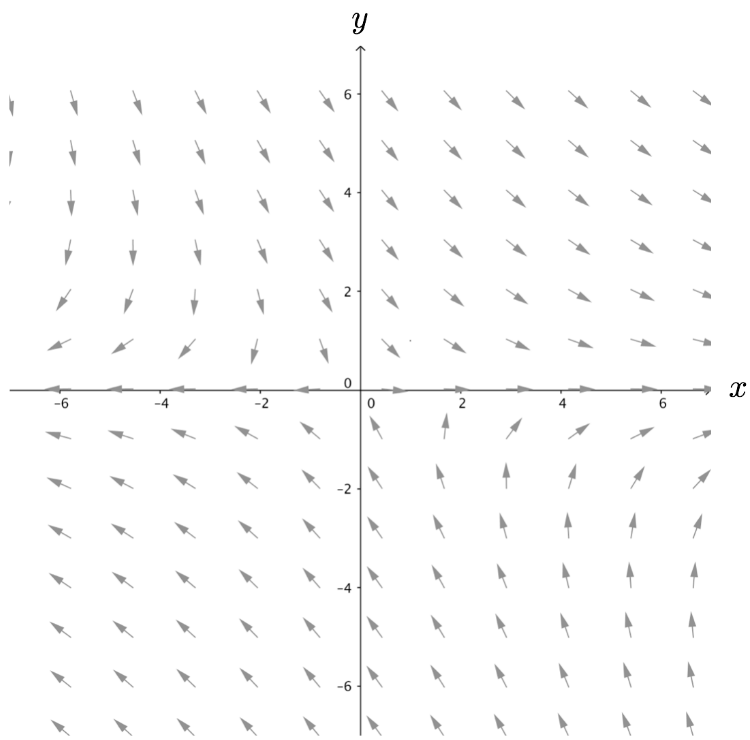
$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

   (c) Explain in words why having two different straight line solutions is useful for finding the $x(t)$ and $y(t)$ equations for any initial condition.

4. Below is a vector field for the system of differential equations:

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = -4y$$

Straight line solutions lie along the line $y = 0$ (with positive exponent in the $x(t)$ and $y(t)$ equations) and along the line $y = -2x$ (with negative exponent in the $x(t)$ and $y(t)$ equations).



(a) Consider two different initial conditions, one at the point (1, 0) and one at the point (3, 0). Determine, with reasons, what happens to the graphs of the two solutions with these initial conditions as time progresses.

(b) Repeat problem 4a for the initial conditions (-1, 2) and (-3, 6).

5. Find a value or a range of values for the parameter n between -4 and 4 (including non-integer values) in the system of differential equations

$$\frac{dx}{dt} = -3x + ny$$
$$\frac{dy}{dt} = 6x + y$$

so that when you view solutions in the x-y plane there are

(a) exactly two different straight line solutions

(b) no straight line solutions

(c) exactly one straight line solution

(d) an infinite number of equilibrium solutions and an infinite number of straight line solutions

6. Consider the following system of differential equations:

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = 2y$$

(a) Without using technology, sketch many different solutions in the phase plane. Explain your reasoning.

(b) Unlike other systems of differential equations that we have been studying, this system can be solved using techniques from our study of 1-dimensional systems. What makes this system different?

(c) Find the general solution in two ways, one using separation of variables and the other using straight line techniques.

(d) Explain how the general solution can help you make sense of the solution graphs in the phase plane.

7. Without using technology, sketch many different solutions in the phase plane for the following system of differential equations. Explain your reasoning. [*Hint*: how many equilibrium solutions are there?]

$$\frac{dx}{dt} = -3x - \frac{1}{2}y$$
$$\frac{dy}{dt} = 6x + y$$

8. **A Swaying Skyscraper**: The following system of rate of change equations is a model for helping us make predictions about the motion of a tall building.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - y + x^3$$

In this simplified system of rate of change equations, x stands for the amount of displacement of the building from the vertical position at any time t and y stands for the horizontal velocity of the building at any time t. Use the GeoGebra Vector Field applet, https://ggbm.at/kkNXUVds, as a tool to explore solutions as viewed in the xy-plane (i.e., the phase plane).

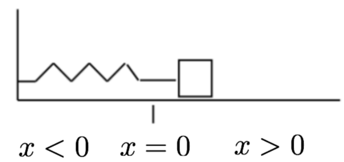
(a) Determine all equilibrium solutions and explain the meaning of each one in terms of the swaying skyscraper. Create any terms needed to classify new types of equilibrium solutions and briefly explain your reasons or imagery behind your choice of terms.

(b) Provide a sketch of several representative curves in the phase plane and give an interpretation for the motion of the building for the different types of curves (e.g., does the building remain standing? If so, for what initial conditions? For what range of initial conditions is a disaster predicted?)

Spiraling Solutions - Spring Mass Revisited

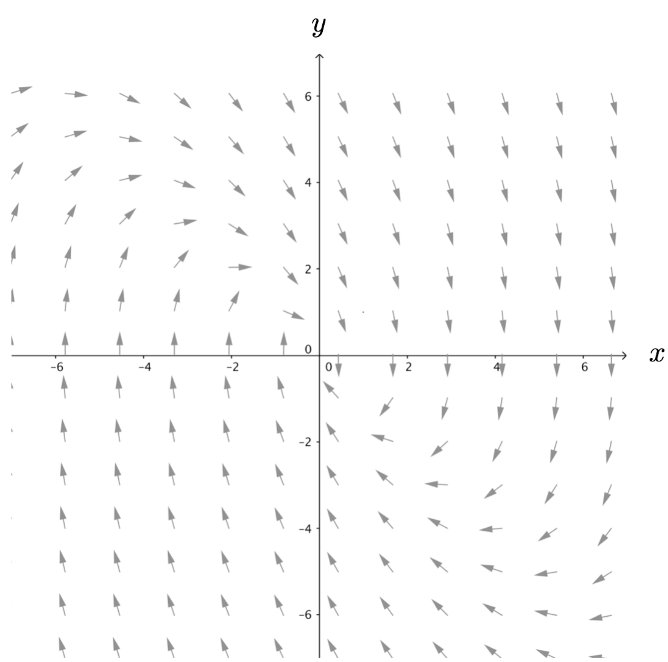
In a previous problem we applied Newton's law of motion for a spring mass system and obtained the second order differential equation $\dfrac{d^2x}{dt^2}+\dfrac{b}{m}\dfrac{dx}{dt}+\dfrac{k}{m}x=0$, where x is the position of the object attached to the end of the spring, m is the mass of the object, b is the damping coefficient, and k is the spring constant. Using the fact that velocity is the derivative of position and choosing the mass $m=1$ and the spring constant $k=2$, we converted this to the following system of two differential equations:

$$\frac{dx}{dt}=y$$
$$\frac{dy}{dt}=-2x-by$$



We were able to figure out the $x(t)$ and $y(t)$ equations when the value of the friction parameter was such that there were straight line solutions in the phase plane. Such a situation is typically referred to as *overdamped*. The situation is called *damped* when the differential equations predict that the mass will oscillate about the 0 position and *undamped* when there is no friction. In the following problems we figure out the $x(t)$ and $y(t)$ equations for the damped. We consider the undamped situation in the homework.

The vector field for the case when $b=2$ is shown below. Based on this vector field, it appears that the differential equations predict that the mass will oscillate back and forth. Even though there are not any straight line solutions, we can still use the same algebraic approach as before to get the $x(t)$ and $y(t)$ equations for any initial condition, but we will have to deal complex numbers. Problems 1-7 outline a way to do this.



1. For the system of differential equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

use the same algebraic approach as before to verify that the slopes of the "straight line" solutions are $-1 \pm i$.

2. For solutions with "straight line" slope $y = (-1+i)x$, find the $x(t)$ and $y(t)$ equations (in terms of complex numbers) for the solution along this "straight line" with initial condition $(1, -1+i)$.

3. For solutions with "straight line" slope $y = (-1 - i)x$, find the $x(t)$ and $y(t)$ equations (in terms of complex numbers) for the solution along this "straight line" with initial condition $(1, -1 - i)$.

4. Use Euler's formula $e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$ to rewrite the $x(t)$ and $y(t)$ equations from problem 2 (call these $x_1(t)$ and $y_1(t)$) and then again from problem 3 (call these $x_2(t)$ and $y_2(t)$).

5. Denise suggests that if you add $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ to $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ the resulting pair of equations is (i) real valued and (ii) a solution to the same system of differential equations. Verify that this is true.

6. Verify that if you subtract $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ from $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ and multiply the result by the complex number i, then the resulting pair of equations will be a real and a solution to the same system of differential equations.

7. (a) Form the general solution to the system of differential equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

(b) What aspect of your general solution could be interpreted as the effect of friction on the spring mass system?

(c) Find the particular solution for the initial condition (2, 3) and sketch the x vs t and y vs t graphs.

Homework Set 11

1. The general solution to

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 2y$$

is

$$x(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$$
$$y(t) = c_1 e^{-t}(-\cos(t) - \sin(t)) + c_2 e^{-t}(-\sin(t) + \cos(t))$$

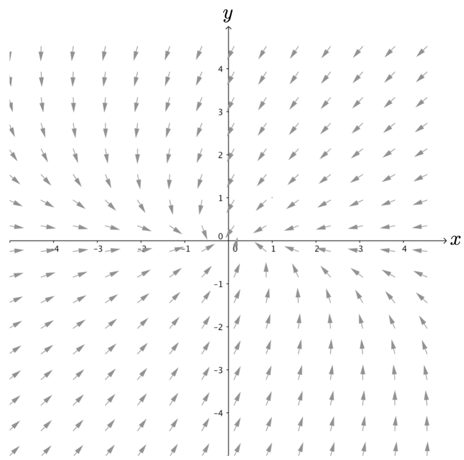
Which part(s) of the general solution accounts for the fact that the differential equations predict that the mass will oscillate about the zero position? Which part(s) of the general solution accounts for the fact that the amplitude of the oscillations decreases over time?

2. Suppose that for a different system of differential equations you got the exact same general solution as homework problem 1 except instead of e^{-t} you got e^{t}. How would this change graphs of solutions in the phase plane? Explain.

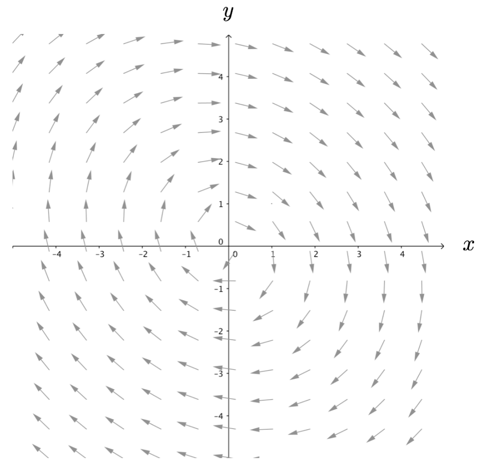
3. Find the general solution to the spring mass problem when there is no friction. Sketch these solution in the phase plane and explain how this general solution fits with your expectation for the behavior of the mass over time. Note: when there is no friction, $b = 0$, and the spring constant $k = 2$, we get

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x$$

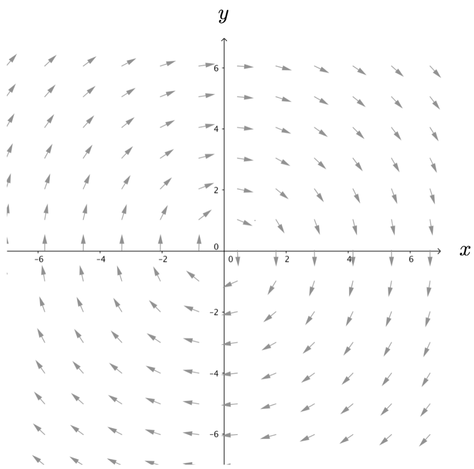
4. Consider the phase planes below:



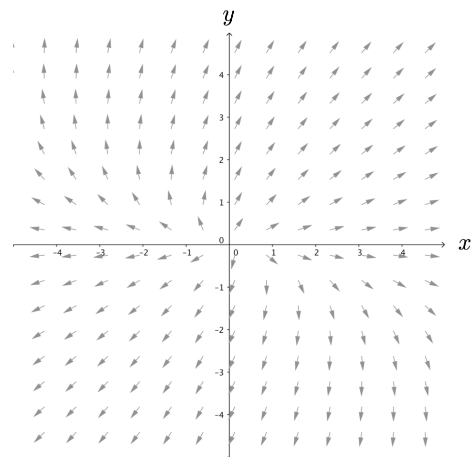
(A)



(B)



(C)



(D)

For each sentence below, fill in the blank with choices from the following two lists:

Spring System (First Blank)	**Solutions (Second Blank)**
a damped spring	$c_1\cos(t) + c_2\sin(t)$
an overdamped spring	$e^{-t}(c_1\cos(t) + c_2\sin(t))$
an undamped spring	$e^{t}(c_1\cos(t) + c_2(\sin(t))$
something other than a spring	$c_1e^{t} + c_2e^{2t}$
	$c_1e^{-t} + c_2e^{-2t}$
	$c_1e^{-t} + c_2e^{2t}$
	$c_1e^{t} + c_2e^{-2t}$

Phase plane (A) corresponds to __________ and the solutions look like $x(t)$= ________________

Phase plane (B) corresponds to __________ and the solutions look like $x(t)$= ________________

Phase plane (C) corresponds to __________ and the solutions look like $x(t)$= ________________

Phase plane (D) corresponds to __________ and the solutions look like $x(t)$= ________________

5. What type of system (undamped, damped, overdamped) do the following best correspond to? Explain your reasoning.
   (a) A car that bounces every time it hits a bump
   (b) A pendulum immersed in a vat of honey
   (c) A bungee jumper
6. In each part, write a differential equation corresponding to the given scenario:
   (a) An undamped spring
   (b) An underdamped spring
   (c) An overdamped spring
7. Does Adding Solutions Always Result in Another Solution?

In deriving the general solution to the spring mass problem, two solutions were added to get another solution. This worked for the particular equations at hand, but does adding two solutions to a system of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

always result in another solution to the same system of differential equations? Below is a proof that this in fact is true.

Claim: If $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions (not necessarily straight line solutions) to a system of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

then the sum of these two solutions is also a solution. That is, if we call the sum of these two solutions $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ where

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

then $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is also a solution to the same system of differential equations.

Proof: In order to show that $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is a solution, we need to verify it satisfies the system of differential equations. This is, we need to show that

$$\begin{aligned} \frac{d}{dt}x_3(t) &= ax_3(t) + by_3(t) \\ \frac{d}{dt}y_3(t) &= cx_3(t) + dy_3(t) \end{aligned}.$$

Since

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

we know that

$$\begin{aligned} \frac{d}{dt}x_3(t) &= \frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t) \\ \frac{d}{dt}y_3(t) &= \frac{d}{dt}y_1(t) + \frac{d}{dt}y_2(t) \end{aligned}. \tag{1}$$

Because $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ is a solution, it satisfies the system of differential equations. That is,

$$\begin{aligned} \frac{d}{dt}x_1(t) &= ax_1(t) + by_1(t) \\ \frac{d}{dt}y_1(t) &= cx_1(t) + dy_1(t) \end{aligned} \tag{2}$$

Similarly, since $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ is a solution,

$$\begin{aligned} \frac{d}{dt}x_2(t) &= ax_2(t) + by_2(t) \\ \frac{d}{dt}y_2(t) &= cx_2(t) + dy_2(t) \end{aligned} \tag{3}$$

Substituting (2) and (3) into (1) yields

$$\begin{aligned} \frac{d}{dt}x_3(t) &= ax_1(t) + by_1(t) + ax_2(t) + by_2(t) \\ \frac{d}{dt}y_3(t) &= cx_1(t) + dy_1(t) + cx_2(t) + dy_2(t) \end{aligned}.$$

Rearranging terms yields

$$\begin{aligned} \frac{d}{dt}x_3(t) &= ax_1(t) + ax_2(t) + by_1(t) + by_2(t) = a[x_1(t) + x_2(t)] + b[y_1(t) + y_2(t)] \\ \frac{d}{dt}y_3(t) &= cx_1(t) + cx_2(t) + dy_1(t) + dy_2(t) = c[x_1(t) + x_2(t)] + d[y_1(t) + y_2(t)] \end{aligned}.$$

Finally, using the fact that

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) + x_2(t) \\ y_1(t) + y_2(t) \end{pmatrix},$$

yields

$$\begin{aligned} \frac{d}{dt}x_3(t) &= ax_3(t) + by_3(t) \\ \frac{d}{dt}y_3(t) &= cx_3(t) + dy_3(t) \end{aligned}$$

which is what we set out to show. Therefore $\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix}$ is also a solution to the system of differential equations.

(a) Suppose that $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by + 1 \\ \frac{dy}{dt} &= cx + dy + 2 \end{aligned}$$

where $a, b, c,$ and d are constants. Josh claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you agree with his claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.

(b) Suppose that $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are solutions to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax^2 + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

where $a, b, c,$ and d are constants. Angela claims that the sum of these two solutions is also a solution to the same system of differential equations. Do you agree with her claim? Either develop a similar proof as above to support this claim or point to where (and why) the above proof fails.

Equilibrium Solutions for Linear Systems

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

1. For each part below, use two different ways (one algebraic and one geometric using nullclines) to figure out the number and location of equilibrium solutions.

   (a) $$\begin{aligned}\frac{dx}{dt} &= 3x + 2y\\ \frac{dy}{dt} &= -2y\end{aligned}$$

   (b) $$\begin{aligned}\frac{dx}{dt} &= 4x - 2y\\ \frac{dy}{dt} &= -2x + y\end{aligned}$$

2. Is it possible to find values of a, b, c, d such that the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

   has <u>exactly</u> two equilibrium solutions? Explain why or why not.

3. Develop criteria (in terms of the parameters $a, b, c,$ and d) that tell us about the number and location of equilibrium solutions for systems of differential equations of the form

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}.$$

Matrix Notation and Equilibrium Solutions for Linear Systems

$$\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

One way to approach problem 3 is to think about there being an infinite number of equilibrium solutions when the two nullclines coincide. That is, when the equations $\begin{matrix}0 = ax + by\\ 0 = cx + dy\end{matrix}$ determine the same set of points. Put another way, the equations are *dependent* when $y = -\frac{a}{b}x$ and $y = -\frac{c}{d}x$ are the same equation. Thus, $-\frac{a}{b} = -\frac{c}{d}$, which says that $-ad = -cb$. Rewriting this yields $ad - bc = 0$.
As shown next, another way to arrive at this result is to use matrix notation and the fact that two equations are dependent when the determinant of the matrix is zero.

$$\begin{aligned}ax + by &= 0\\ cx + dy &= 0\end{aligned} \Longrightarrow \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}$$

Thus, the equations $\begin{aligned}ax + by &= 0\\ cx + dy &= 0\end{aligned}$ are dependent when the determinant of the coefficient matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is zero. That is, when $ad - bc = 0$.

Next, we develop an approach for finding the general solution to a system of differential equations of the form $\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\end{aligned}$ by first finding the value of the exponent (that is, the **eigenvalue**) associated with any straight line solution *before* finding the slope of the straight line solutions (typically called **eigensolutions**). Note that in your previous work you first found the slope of straight line solutions and then found the exponent. Some students have referred to this as the "slope first" method. In the pages that follow, an alternative approach is developed the "eigenvalue first" method.

We develop this alternative method for four reasons:

- The eigenvalue first method can be used for systems of three or more differential equations whereas the slope first method cannot.
- Oftentimes just knowing the eigenvalues is sufficient for understanding the overall picture of solutions in the phase plane and so therefore this method is more efficient.
- The eigenvalue first approach makes important connections with linear algebra.
- The eigenvalue first approach is algebraically more efficient.

Eigenvalue First Method

For linear systems of the form $\begin{array}{rcl} \frac{dx}{dt} & = & ax+by \\ \frac{dy}{dt} & = & cx+dy \end{array}$, one way to determine the exponent (i.e. λ, the eigenvalue) for possible straight line solutions (or eigensolutions) is to use the fact that if eigensolutions exist in the phase plane, then $\frac{dx}{dt} = \lambda x$ and $\frac{dy}{dt} = \lambda y$.

4. Explain why this has to be true.

Combining the fact that $\begin{array}{rcl} \frac{dx}{dt} & = & ax+by \\ \frac{dy}{dt} & = & cx+dy \end{array}$ with the fact that for straight line solutions $\frac{dx}{dt} = \lambda x$ and $\frac{dy}{dt} = \lambda y$ along the straight line, we can set up the following two equations:

$$\begin{aligned} ax+by &= \lambda x \\ cx+dy &= \lambda y \end{aligned} \tag{1}$$

Rearranging these equations we get

$$\begin{aligned} (a-\lambda)x+by &= 0 \\ cx+(d-\lambda)y &= 0 \end{aligned} \tag{2}$$

Note that although these equations look similar to the nullcline equations, the coefficients are different.

If the equations from (2) are dependent then you get straight line solutions with a particular value of λ corresponding to an exponent from the straight line solution.

5. Explain why this has to be true.

Rewriting these dependent equations in slope form yields $y = -\frac{a-\lambda}{b}x$ and $y = -\frac{c}{d-\lambda}x$ and thus $-\frac{a-\lambda}{b} = -\frac{c}{d-\lambda}$. Rearranging this last equation we get the following:

$$\begin{aligned}
&(a-\lambda)(d-\lambda) - bc = 0 \\
\Rightarrow &\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \\
\Rightarrow &\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}
\end{aligned}$$

We can more efficiently obtain this same result using matrix notation and the fact that two equations are dependent when the determinant of the coefficient matrix is zero as follows:

$$\begin{aligned} (a-\lambda) + by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned} \Longrightarrow \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus, the equations $\begin{aligned} (a-\lambda) + by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned}$ are dependent when the determinant of the coefficient matrix

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

is zero. That is, when $(a-\lambda)(d-\lambda) - bc = 0$.

EXAMPLE:

Determine the general solution for the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y \\ \frac{dy}{dt} &= x + 3y \end{aligned}$$

using the "eigenvalue first" approach.

In order to get eigensolutions, we need to have

$$\begin{aligned}
&\begin{aligned} 4x + 2y &= \lambda x \\ x + 3y &= \lambda y \end{aligned} \\
\Rightarrow &\begin{aligned} (4-\lambda)x + 2y &= 0 \\ x + (3-\lambda)y &= 0 \end{aligned} \\
\Rightarrow &\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow &(4-\lambda)(3-\lambda) - 2 = 0 \\
\Rightarrow &\lambda^2 - 7\lambda + 10 = 0 \\
\Rightarrow &(\lambda-5)(\lambda-2) = 0 \\
\Rightarrow &\lambda = 2, \lambda = 5
\end{aligned}$$

<u>For $\lambda = 2$</u>

Since these two equations $\begin{aligned} 4x + 2y &= 2x \\ x + 3y &= 2y \end{aligned}$ are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line $y = -x$.

Any solution along this line can therefore be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda = 5$

Since these two equations $\begin{aligned} 4x + 2y &= 5x \\ x + 3y &= 5y \end{aligned}$ are dependent, we can use either one to determine the straight line of vectors (called eigenvectors) in the phase plane. In this case, straight line solutions are found along the line $y = \frac{1}{2}x$.

Any solution along this line can therefore be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

6. In the previous example the general solution was determined to be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

What is the specific solution for the initial condition $(-3, -2)$? Without using technology, sketch the graph of this solution in the phase plane (for $t \to \infty$ and as $t \to -\infty$) and explain how you figured out what the graph looks like based on the equations for the solution.

Homework Set 12

1. If the general solution for a system of differential equations of the form
$$\begin{aligned}\frac{dx}{dt} &= ax+by\\ \frac{dy}{dt} &= cx+dy\end{aligned}$$
is
$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = k_1e^{-2t}\begin{pmatrix}-1\\4\end{pmatrix} + k_2e^{-t}\begin{pmatrix}2\\1\end{pmatrix},$$
what do solutions in phase plane look like? What do solutions that are not straight lines look like? Do they curve a particular way? Figure out a way to use the general solution (without technology) to decide. Explain and graph your ideas.

2. Repeat problem 1 for the general solution
$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = k_1e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + k_2e^{3t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

3. Repeat problem 1 for the general solution
$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = k_1e^{0t}\begin{pmatrix}1\\2\end{pmatrix} + k_2e^{-2t}\begin{pmatrix}-3\\2\end{pmatrix}.$$

4. For each of the following systems of differential equations, find the general solution and then sketch the **phase portrait** (i.e. graphs of solutions viewed in the phase plane) <u>without</u> using technology.

   (a) $\begin{aligned}\frac{dx}{dt} &= 2x+y\\ \frac{dy}{dt} &= x+y\end{aligned}$ (b) $\begin{aligned}\frac{dx}{dt} &= -4x-2y\\ \frac{dy}{dt} &= -x-3y\end{aligned}$ (c) $\begin{aligned}\frac{dx}{dt} &= 4x+2y\\ \frac{dy}{dt} &= x+3y\end{aligned}$

   (d) $\begin{aligned}\frac{dx}{dt} &= 4x-2y\\ \frac{dy}{dt} &= -2x+y\end{aligned}$ (e) $\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -4x-y\end{aligned}$ (f) $\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= 2x-y\end{aligned}$

   (g) $\begin{aligned}\frac{dx}{dt} &= 2x\\ \frac{dy}{dt} &= 2y\end{aligned}$ (h) $\begin{aligned}\frac{dx}{dt} &= -3x-\tfrac{1}{2}y\\ \frac{dy}{dt} &= 6x+y\end{aligned}$

5. For the system of differential equations
$$\begin{aligned}\frac{dx}{dt} &= rx+2y\\ \frac{dy}{dt} &= 3x+ry\end{aligned}$$
figure out all the possible types of equilibrium solutions for different values of r, where r is some real number. Show all work to support your conclusions.

6. Denise claims that all solutions (except the equilibrium solution) to the system of differential equations
$$\begin{aligned}\frac{dx}{dt} &= ax+by\\ \frac{dy}{dt} &= cx+dy\end{aligned}$$
will spiral if $(ad-bc)$ is negative. Do you are agree with Denise's claim? If yes, justify your response. If not, explain why not.

7. Consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= px + qy \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

   (a) Explain why if the eigenvalues are distinct real numbers, the general form of the solution can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\phi_1 + c_2 e^{\lambda_2 t}\phi_2$$

   where ϕ_1 and ϕ_2 are the eigenvectors associated with λ_1 and λ_2.

   (b) Explain why if the eigenvalues are complex numbers of the form $a \pm bi$ then the general solution is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{at} v$$

   where v is a vector containing arbitrary constants c_1 and c_2 and other terms involving $\sin(t)$ and $\cos(t)$.

   (c) Complete the following table by reflecting on and organizing what you've figured out about the phase portrait for systems of linear differential equations based on knowing just the eigenvalues.

Eigenvalues	Typical phase portrait	Basic format of the general solution
two distinct positive real numbers		
one positive and one negative real number		
two distinct negative real numbers		
two identical (repeated) positive real numbers		
a complex conjugate pair with negative real part		
a complex conjugate pair with positive real part		
a complex conjugate pair with no real part		

Second Order Linear Differential Equations

A second order linear differential equation has the form

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)$$

where P, Q, R, and G are continuous functions. There are many applications for which this type of differential equation is a useful model. Your previous work with the spring mass problem was one such example. Here are some other examples.

Glass Breaking: You probably have all seen in cartoons or on Mythbusters where a wineglass is broken by singing a particular high-pitched note. The phenomenon that makes this possible is called *resonance*. Resonance results from the fact that the crystalline structures of certain solids have natural frequencies of vibration. An external force of the same frequency will "resonate" with the object and create a huge increase in energy. For instance, if the frequency of a musical note matches the natural vibration of a crystal wineglass, the glass will vibrate with increasing amplitude until it shatters. The following is one model for understanding resonance:

$$\frac{d^2x}{dt^2} + k^2x = \cos(kt)$$

Tacoma Narrows Bridge: The Tacoma Narrows Bridge in Washington State was one of the largest suspended bridges built at the time. The bridge connecting the Tacoma Narrows channel collapsed in a dramatic way on Thursday November 7, 1940. Winds of 35-46 miles/hours produced an oscillation which eventually broke the construction. The bridge began first to vibrate torsionally, giving it a twisting motion. Later the vibrations entered a natural resonance (same term as in the glass breaking) with the bridge. Here is a simplified second order differential equation that models the situation of the Tacoma Bridge:

$$\frac{d^2y}{dt^2} + 4y = 2\sin(2.1t)$$

Sometimes resonance is a good thing! Violins, for instance, are designed so that their body resonates at as many different frequencies as possible, which allows you to hear the vibrations of the strings!

There are many other situations that can be modeled with second order differential equations, including RLC circuits, pendulums, car springs bouncing, etc. In this section you will learn how to solve second order linear differential equations with constant coefficients. That is, equations where P, Q, and R are constant. If G is zero, then the equation is called **homogeneous**. When G is nonzero then the equation is called **nonhomogeneous**. As you will discover in the problems that follow, the distinction between homogeneous and non-homogeneous equations will be quite useful.

Guess and Test

1. (a) Read the following equations *with meaning*, by completing the following sentence, "$x(t)$ is a function for which its second derivative ..." (try saying "itself" instead of "x").

   i. $\dfrac{d^2x}{dt^2} = -x$ ii. $\dfrac{d^2x}{dt^2} + x = 0$ iii. $\dfrac{d^2x}{dt^2} + 4x = 0$ iv. $\dfrac{d^2x}{dt^2} = x$

   (b) For each differential equation above, based on your readings *with meaning*, find two different solution functions.

2. Your task in this problem is to use the "guess and test" approach to find a solution to the linear second order, homogeneous differential equation

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0$$

   By now you know very well that solutions are functions. What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to zero? Explain briefly the rationale for your guess and then test it out to see if it works. If it doesn't work keep trying.

3. Determine if a constant multiple of your solution is also a solution.

4. Try and find a different solution, one that is not a constant multiple of your solution to problem 2.

5. Determine the *general solution* to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 0$.

6. Consider again the differential equation $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 0$.

By guessing $x(t) = e^{rt}$, show that this guess yields a solution to the differential equation precisely when $r^2+10r+9 = 0$.	
Solve this quadratic equation to find two different values of r.	
State two different solutions for the differential equation, one for each value of r.	
Form the general solution by multiplying your two solutions by constants c_1 and c_2, and adding the results.	
Congratulate yourself :)	

7. Find the general solution to the following differential equation: $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - 6x = 0$.

The Nonhomogeneous Case

8. In this next problem your task is to find a solution to the following **nonhomogeneous** version of the differential equation from the first problem:

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18.$$

What is your best guess for a function whose second derivative plus 10 times its first derivative plus 9 times the function itself sum to 18? Test out your guess to see if it works. If it doesn't work keep trying.

The solution you found in the previous problem is called the **particular solution** to the nonhomogeneous differential equation. To find the general solution to the nonhomogeneous differential equation you simply add the particular solution to the general solution to the corresponding homogeneous equation. This 3-step strategy (1 - Find the general solution to the corresponding homogeneous equation; 2 - Find the particular solution to the nonhomogeneous equation, 3 - Add the previous results) is called the **Method of Undetermined Coefficients**.

9. Write down the general solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 18$ and give a convincing argument for why this sum is in fact a solution is the nonhomogeneous differential equation.

10. Sean and Phil are trying to find the particular solution to $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 85\sin(2t)$. Sean guesses $x(t) = A\sin(2t)$ for the particular solution and Phil guesses $x(t) = B\cos(2t)$.

    (a) Do you think these are reasonable guesses? Explain why or why not.

    (b) For each of their guesses, can you find a value of A or B such that their guess is a solution? If yes, write down the general solution. If no, come up with a different guess for the particular solution and show that your guess is correct.

11. Write down the general solution to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 85\sin(2t)$.

12. Find the general solution to $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 9x = 85\sin(2t)+18$. Explain why you can do this by combining results from the previous problems.

13. An aside on complex numbers:

 (a) Show that $x(s) = e^{is}$ and $x(s) = \cos(s) + i\sin(s)$ are both solutions to the differential equation $dx/ds = ix$ with $x(0) = 1$. What does the uniqueness theorem imply about these two solutions?

 (b) The above result is called Euler's formula. Multiplying by $e^{\alpha t}$ and using $s = \beta t$, we can rewrite the formula into the following form: $e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$. Use this to find a similar formula for $e^{(\alpha-\beta i)t}$.

(c) Suppose you have two functions:

$$A(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$
$$B(t) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

Simplify the following expressions in (i) and (ii) then answer (iii) and (iv).

i. $x_1(t) = \dfrac{A(t) + B(t)}{2}$

ii. $x_2(t) = i\dfrac{A(t) - B(t)}{2}$

iii. What do you notice about your solutions in (i) and (ii), compared to $A(t)$ and $B(t)$?

iv. If $A(t)$ and $B(t)$ were solutions to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0,$$

would $x_1(t)$ and $x_2(t)$ be solutions too? How about $c_1x_1(t) + c_2x_2(t)$ for arbitrary constants c_1 and c_2?

14. Find the general solution to the homogeneous differential equation

$$\frac{d^2x}{dt^2} + 25x = 0$$

You will find that your guess results in complex roots to the quadratic. Use the above results on exponentiation of complex numbers to find the general solution to the differential equation.

15. (a) Consider the nonhomogeneous differential equation

$$\frac{d^2x}{dt^2} + 25x = 10\cos(5t)$$

Suppose you wish to find the particular solution to this differential equation. Explain why a guess of the form $x(t) = A\cos(5t) + B\sin(5t)$ is doomed to fail.

(b) Nevertheless, explain why your particular solution must have terms that *look like* $\cos(5t)$ and $\sin(5t)$.

(c) For an unknown differentiable function $f(t)$, write down the first and second derivatives of $tf(t)$, what do you notice?

(d) Explain why a guess of $At\cos(5t)$ is insufficient to find the particular solution.

(e) Use the guess $x(t) = t(A\cos(5t) + B\sin(5t))$ to find a particular solution to the above equation.

16. (a) Find the general solution to

$$\frac{d^2x}{dt^2} + 25x = 10\cos(5t).$$

(b) Find the specific solution for initial conditions $x(0) = 0$, $x'(0) = 1$.

Homework Set 13

1. When we are solving a nonhomogeneous second order linear differential equation, the above task sequence had you create a general strategy to first find the solution to the corresponding homogenous equation. You may or may not have found that you always end up solving a quadratic equation to find the coefficients of the exponent variable. In other words, the equation looks like this.

$$k^2 + bk + c = 0$$

   This is called the **characteristic equation** for the homogeneous linear DE. Find the characteristic equation and solve to find the general solution for the following homogeneous linear differential equations.

   (a) $y'' + y' + 12y = 0$
   (b) $y'' + y' + y = 0$
   (c) $y'' + 9y = 0$

2. Find the solution to the following linear second order differential equations.

   (a) $y'' - 4y' = 0$
   (b) $y'' - 4y' = x$
   (c) $y'' - 4y' = x + \sin(x)$
   (d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2x + 3$
   (e) $y'' - 5y' + 4y = e^{5x}$
   (f) $y'' - 5y' + 4y = e^{4x}$

3. Find the solution to the initial value problem

$$y'' + 2y' + 2y = 0 \quad \text{where} \quad y(\pi/4) = 2 \quad \text{and} \quad y'(\pi/4) = -2.$$

4. Create a table that provides the guess you might make for the particular solution of a second order DE when you are faced with different possible right hand sides of your DE. For example, if the right hand side is general $A\cos(kt)$, what would you guess... etc.

5. In everyday life resonance can be a fairly common phenomenon although you may not realize it. Resonance occurs when a system is forced at its natural frequency, leading to a build-up of the amplitude of oscillation and energy. The effect is familiar to most as the high pitched squeal over a PA system caused by microphone feedback. Mathematically, resonance can be seen as a nonhomogeneous second order differential equation whose particular solution is of the same form as the complementary function. To see how this happens, find the general solution to this differential equation:

$$\frac{d^2y}{dt^2} + 4y = 3\cos(2t)$$

6. In this question we will interpret the equation $\frac{d^2y}{dt^2} + 4y = 3\cos(2t)$ as an undamped spring-mass system being periodically driven by the force $F(t) = 3\cos(2t)$.

   (a) Explain why one should expect the spring to eventually break.

   (b) Explore the results of adding a small amount of friction to the system. (*Hint*: the new system would be $\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 4y = 3\cos(2t)$, $b > 0$)

7. The Tacoma Narrows Bridge in Washington State was one of the largest suspended bridges built at the time. The bridge connecting the Tacoma Narrows channel collapsed in a dramatic way on Thursday November 7, 1940. Winds of 35-46 miles/hours produced an oscillation which eventually broke the construction. The bridge began first to vibrate torsinonally, giving it a twisting motion. Later the vibrations entered a natural resonance (same term as in the glass breaking) with the bridge. Here is a simplified second order differential equation that models the situation of the Tacoma Bridge:

$$\frac{d^2y}{dt^2} + 4y = 2\sin(2.1t)$$

   Solve this differential equation and interpret your solution.

8. Suppose you are solving a DE of the following form:

$$y'' + by' + cy = A\sin(mt)$$

   Determine the parameters of m that would assure you that you can use a particular solution guess of

$$y_p = A\sin(mt) + B\cos(mt)$$

   And not

$$y_p = t(A\sin(mt) + B\cos(mt))$$

   Explain your answer.

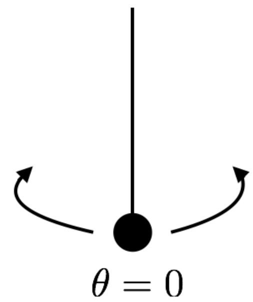
9. Use the Internet (or if you are feeling old school, a book) to learn about the technique of **variation of parameters**, and use it to solve the following two differential equations. In each case, compare your solution with the one you would get through the method of undetermined coefficients.

   (a) $y'' - y = e^{2t}$

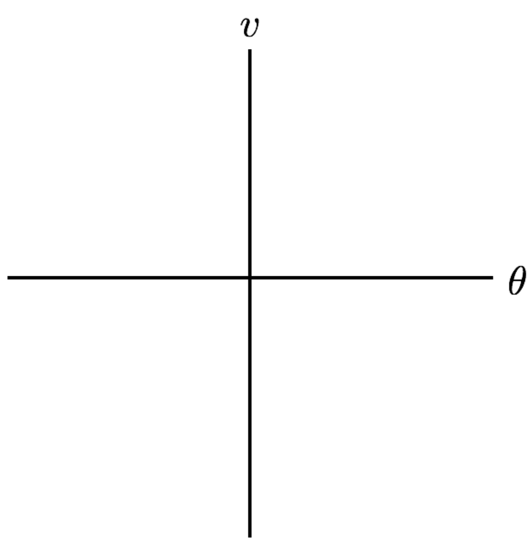
   (b) $y'' + y = \cos(t)$

In the Swing of Things

A pendulum is attached to a wall in such a way that it is free to rotate around in a complete circle. Without provocation, Debra takes a baseball bat and hits it, giving it an initial velocity and setting it in motion.



1. If we call θ the angular position of the pendulum (where $\theta = 0$ corresponds to when the pendulum is hanging straight down) and we call the velocity of the pendulum v, what would angular position versus velocity graphs look like for a variety of different initial velocities due to Debra's hit? Provide a brief description of the motion of the pendulum for your graphs.



2. How many equilibrium solutions are there, where are they, and how would you classify them?

Applying Newton's 2nd Law of motion (where $\theta = 0$ corresponds to the downward vertical position and counterclockwise corresponds to positive angles θ) yields the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0$$

where b is the coefficient of damping, m is the mass of the pendulum, g is the gravity constant, and l is the length of the pendulum (See homework problem 5 for a derivation of this equation). Estimating the parameter values for the pendulum that Debra hits and changing this second order differential equation to a system of differential equations yields

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -0.2v - \sin(\theta)\end{aligned}$$

3. How many equilibrium solutions does this system of differential equations have, where are they, and based on the context what types of equilibrium solutions would you expect them to be? How does this connect with your answer to 2?

4. You might recall that if θ is small, $\sin(\theta) \approx \theta$. Explain why this is true and then use this fact to approximate the above system with a linear system and classify the equilibrium solution at the origin.

5. Classify the equilibrium point at $\theta = \pi$.

6. Use the GeoGebra applet, https://ggbm.at/SpfDSc5Q, to approximate the range of initial velocities with zero initial displacement that will result in the pendulum making exactly one complete rotation before eventually coming to rest.

Linearization and Linear Stability Analysis

In the next several questions we will develop tools to analyze equilibria of nonlinear systems. To do this, we will first build our intuition by studying first order nonlinear equations.

7. Recall from Calculus that the linearization, $L(h)$, of a function around a point of interest, x^*, is given by $L(h) \equiv f(x^*) + hf'(x^*)$. The key feature of the linearization is that, when $x \approx x^*$, that is, $x = x^* + h$ for $h \approx 0$, then $f(x) \approx L(h)$.

Find the linearization of $f(x) = 1 - x^2$ around $x^* = 1$.	
If $x \approx 1$, x can be written as $x = 1 + h$ where $h \approx 0$. Suppose x follows the differential equation $\frac{dx}{dt} = 1 - x^2$. Use the linearization above to write down a linear differential equation for $\frac{dh}{dt}$.	
According to the above differential equation, what is the long term behavior of h?	
If $x(0) \approx 1$, what does the long term behavior of h tell you about the long term behavior of x?	

8. (a) Consider again $\dfrac{dx}{dt} = 1 - x^2$, but this time with $x(0) \approx -1$. Find a new linearization and use it to make a long term prediction about x.

(b) Why was it necessary to construct a **new** linearization to study $x(0) \approx -1$?

(c) Using linearization to determine the stability of a critical point is called "linear stability analysis." Use a phase line to corroborate your linear stability analysis.

(d) For an arbitrary system, $\dfrac{dx}{dt} = f(x)$ with an equilibrium point at $x = x^*$, describe how you can use linear stability analysis to determine the stability of the equilibrium point.

9. Consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= 1 - x^2 \\ \frac{dy}{dt} &= -3x - 3y\end{aligned}$$

(a) Algebraically find the equilibrium solutions.

(b) Tanesha used the GeoGebra Vector field applet, https://ggbm.at/kkNXUVds, to plot the vector field associated with the differential equation. Based on this vector field, how would you classify the equilibria?

We can also perform linear stability analysis on a system of two or more variables, such as the one in the previous problem. Consider a function $f(x, y)$, then Taylor's theorem states that, if $(x, y) \approx (x^*, y^*)$, that is, if $(x, y) = (x^* + h_1, y^* + h_2)$ where $h_1 \approx 0$ and $h_2 \approx 0$, then

$$f(x, y) \approx L(h_1, h_2) = f(x^*, y^*) + h_1 f_x(x^*, y^*) + h_2 f_y(x^*, y^*)$$

where f_x and f_y are the partial derivatives of f with respect to x and y, respectively.

$L(h_1, h_2)$ is called the linearization of $f(x, y)$ around (x^*, y^*).

10. Consider the system of differential equations:

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Let's first study the equilibrium at (1,-1).

If the system had $x(0) \approx 1$ and $y(0) \approx -1$, we could write $x = 1 + h_1$ and $y = -1 + h_2$, with $h_1 \approx 0$ and $h_2 \approx 0$. Use the linearization of the original system of equations around $(1, -1)$ to write down a system of differential equations for h_1 and h_2	
What are the long term behaviors of h_1 and h_2?	
What can you conclude about the long term behaviors of x and y?	
Classify the equilibrium point $(1, -1)$, according to your linear stability analysis.	

11. (a) Consider again

$$\frac{dx}{dt} = 1 - x^2$$
$$\frac{dy}{dt} = -3x - 3y$$

Use linear stability analysis to classify the equilibrium point at $(-1, 1)$.

(b) Combine your results from question 10 and 11a, to sketch a possible phase plane for the system of differential equations. Does an analysis of the system using nullclines corroborate your linear stability analysis?

12. For a system of differential equations

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with an equilibrium point at (x^*, y^*), the matrix

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

is called the **Jacobian matrix**. Explain how you can use the Jacobian matrix to determine the behavior of a the system of differential equations near (x^*, y^*).

13. Use linear stability analysis to classify the critical points you found in the pendulum system.

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -0.2v - \sin(\theta)$$

Homework Set 14

1. Bees and Flowers II. In an earlier problem, we studied systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is both species are harmed by interaction) or cooperative (that is both species benefit from interaction).

$$\begin{array}{ll} \text{(A)} & \text{(B)} \\ \dfrac{dx}{dt} = -5x + 2xy & \dfrac{dx}{dt} = 3x(1 - \dfrac{x}{3}) - \dfrac{1}{10}xy \\ \dfrac{dy}{dt} = -4y + 3xy & \dfrac{dy}{dt} = 2y(1 - \dfrac{y}{10}) - \dfrac{1}{5}xy \end{array}$$

   (a) Explain why the second system of rate of change equations describes a situation where the two species are competitive.

   (b) Verify that the equilibrium solutions for system (B) are (0,0), (3, 0), (0, 10), and $(\frac{20}{9}, \frac{70}{9})$.

   (c) Determine the linearized system of differential equations about each equilibrium solution and use the information you gain about the solutions near each of these equilibrium solutions to sketch the phase portrait.

2. Without using technology, use the tools of linearization and nullclines to sketch the phase portrait for the nonlinear system:

$$\frac{dx}{dt} = \cos(y)$$
$$\frac{dy}{dt} = y - x$$

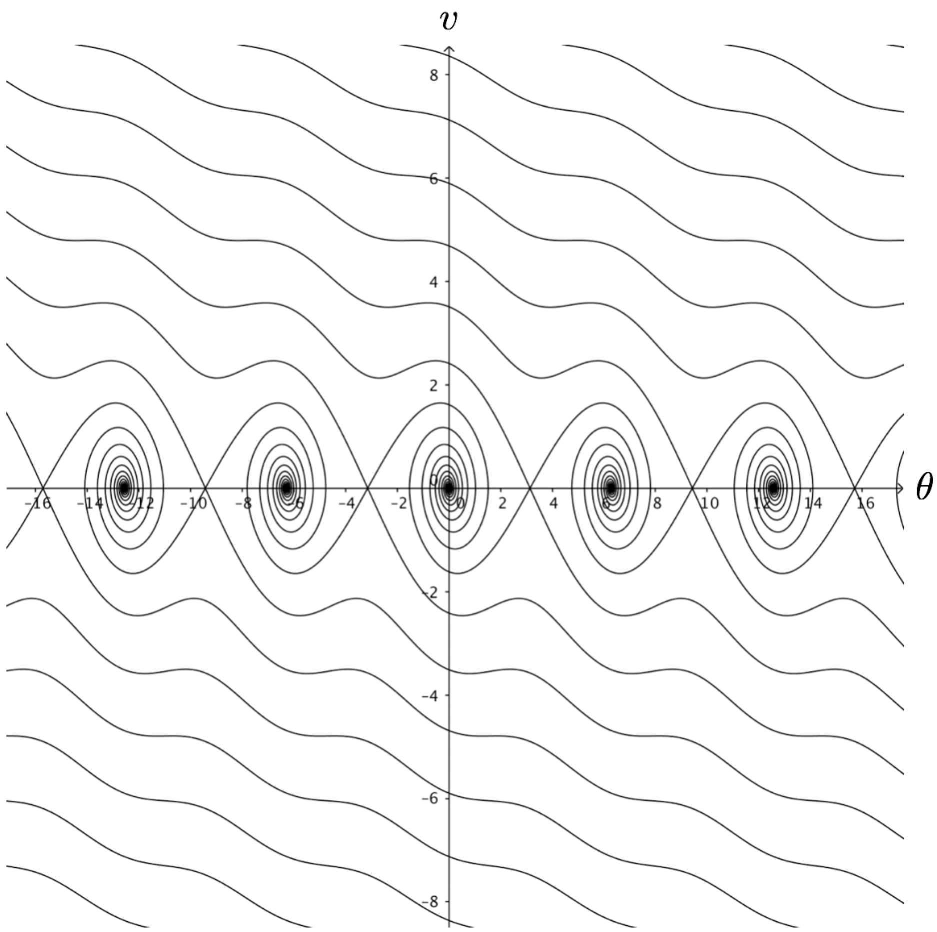
   Be as accurate as possible and show all supporting work.

3. When the John Hancock Building in Boston, MA was first built it tended to sway back and forth so much so that people in the top floors experienced motion sickness. Similar to the spring mass system, we can model the back and forth motion of the building by adding a gravity term to the spring mass model.

   The following system of rate of change equations is a model for helping us make predictions about the motion of a skyscraper swaying in the wind. In this simplified system of rate of change equations, x is the amount of displacement of the building from the vertical position at any time t and y is the horizontal velocity of the building at any time t. Use what you know about linear stability analysis to analyze the behavior of the systems at the critical points and compare to your earlier work. (You might want to use a GeoGebra vector field applet, https://ggbm.at/kkNXUVds, to help understand it first).

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - y + x^3$$

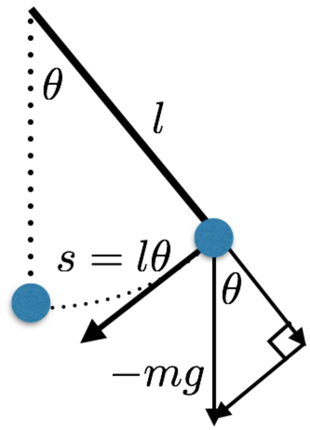
4. Consider the phase plane below for the damped pendulum:



(a) Shade in the region(s) corresponding to initial conditions that will make one full revolution before coming to a stop.

(b) Use a different shading to show the region(s) corresponding to initial conditions that will make two full revolutions before coming to a stop.

5. Consider the diagram below for the pendulum:



(a) The force, due to gravity, on the bob of the pendulum is given by $-mg$. Explain why the proportion of that gravitational force, in the direction tangent to the path of the pendulum's bob, is given by $F = -mg\sin(\theta)$.

(b) In the diagram above, explain why the length of the dotted arc is given by $s = l\theta$, when θ is measured in radians.

(c) The frictional force (due to friction at the fixed point of the pendulum, or due to air resistance, or a combination of these two) opposes the motion of the pendulum. Carefully explain why this force can be represented as $F = -b\dfrac{ds}{dt} = -bl\dfrac{d\theta}{dt}$.

(d) Newton's law states that force is given by mass times acceleration. If m is the mass of the pendulum bob, explain why $F = m\dfrac{d^2s}{dt^2} = ml\dfrac{d^2\theta}{dt^2}$.

(e) Explain how the previous parts of this question can be combined to arrive at a differential equation:

$$ml\frac{d^2\theta}{dt^2} = -bl\frac{d\theta}{dt} - mg\sin(\theta)$$

(f) By defining $v = \dfrac{d\theta}{dt}$, develop a pair of first order differential equations for the (θ, v) system.

Glossary for First Order Linear Differential Equations

Analytic approach: In this course, use have two analytic approaches **separation of variables** and the technique for **first order linear differential equations**. These approaches provide either general or particular solutions in algebraic or analytic form.

Autonomous differential equation: A differential equation where the derivative is dependent only on the dependent variable. For example $\frac{dy}{dt} = 2y - 3$ is autonomous, but $\frac{dy}{dt} = 2t - 3$ is not autonomous.

Bifurcation diagram: A plot of equilibrium solutions versus a parameter. Additionally, one can show phase lines on the graph which show whether equilibrium solutions are stable (attractor), unstable (repeller), or semi-stable (node).

Bifurcation value: A value of the parameter for which there is a change in the number or type of equilibrium solutions.

Differential equation: A differential equation is also known as a **rate of change equation**. An equation for an unknown function in terms of its derivative. Suppose $y = y(t)$ is some unknown function, then a differential equation, or rate of change equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. **First order** differential equation contains only the first derivative. **Second order** differential equations contains derivatives up to the second derivative. An **ordinary differential equation (ODE)** is a differential equation whose derivatives pertain to only one variable, typically derivatives with respect to time. A **partial differential equation (PDE)** is a differential equation whose derivatives pertain to multiple variables.

Equilibrium solution: A constant function that satisfies a given differential equation. There are three types of equilibrium solutions for first order differential equations: attractors (stable), repellers (unstable), and nodes (semi-stable).

Euler's method: Informally referred to as the "tip to tail" method; this is a numerical method to find approximate solutions to a given differential equation.

Exact solution: A function that satisfies a given differential equation. That is, when the function is inserted into the differential equation a true statement results.

Explicit solution: The general solution has been written so that it is in the form $y(t) = e^{2t}$. Contrast this with **implicit solution**.

First order linear differential equation: A differential equation that can be written in the form $\frac{dy}{dt} + g(t)y = r(t)$, where $g(t)$ and $r(t)$ are both continuous functions. This type of differential equation is solved using the analytic technique of **reverse product rule.**

General solution: An algebraic (sometimes referred to as analytic) representation of the family of functions that solve a given differential equation.

Implicit solution: The general solution has been left in a form that has not been (or cannot be) algebraically solved. For example, $y(t)^5 + y(t) = e^{2t}$.

Initial condition or initial value: A specific point through which the solution to a differential equation will pass. Usually expressed as $y_{t_0} = y_0$. For example, $y(0) = 2$ (or $y(2) = 6$) could be an initial condition that is then used to determine the **particular solution** from the **general solution**.

Initial value problem (IVP): A differential equation together with an initial condition (initial value) is called an Initial Value Problem (IVP).

Integrating factor: See **reverse product rule**.

Numerical approach: Provides numerical approximations to an initial value problem. One such method is **Euler's method**. Other methods include the Improved Euler's method and the **Runge-Kutta** method.

Particular solution: An algebraic (or analytic) representation of a specific function that solves the differential equation and contains a specified point, usually called the **initial value**. A differential equation together with an initial condition is referred to as an **initial value problem**.

Qualitative / graphical approach An approach to solving a differential equation that considers slopes and how the solution follows the slopes in a field.

Reverse product rule: A technique for solving a **first order linear differential equation** by introducing an unknown function u to help ?undo? the product rule. u is sometimes called an **integrating factor**.

Runge-Kutta (RK4) method: A fourth order method used in solving differential equations numerically. Contrast with **Euler's method** which is first order.

Separable differential equation: Differential equation that can be written in the form $\frac{dy}{dx} = f(y)g(x)$ and, when possible, solved using the analytic technique of **separation of variables**.

Separation of variables: An analytic technique to solve a differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ by separating the variables (i.e., by rewriting it as $\frac{dy}{f(y)} = g(x)dx$) and integrating both sides if possible.

Slope field: A graphical representation of the slopes at many different points in a coordinate plane where each slope is determined by the derivative (rate of change) at any point in the plane. Slope fields can be used to sketch in graphs of solution functions. A curve that follows the slopes is the graphical analogue of inserting a function into the differential equation with the result giving a true statement.

Uniqueness theorem: Informally, the terms "unique" or "uniqueness" refers to whether or not two solution functions ever touch or cross each other. Refer to page 5.4 of the materials for the formal theorem.

Glossary for Systems, Second Order, and Nonlinear DEs

Characteristic equation: A polynomial equation corresponding to a second order linear differential equation that is used to help find solutions.

Damping, Overdamped, Undamped: Damping is the presence of a friction-like force in the system. Undamped is the lack of friction-like in the system. A system is called overdamped if the friction-like parameter exceeds a certain value determined by other parameters in the system.

Dependent (pertaining to linear algebraic equations): A homogeneous system of two equations is dependent when it has infinitely many solutions.

Eigensolution: A straight line solution formed from an eigenvalue, eigenvector pair.

Eigenvalue: The value of the exponent associated with any straight line solution to a system of differential equations.

Homogeneous differential equation: The following second order differential equation, $P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=G(t)$, is homogenous when $G(t)=0$. The same holds true for higher order differential equations.

Isocline: An isocline is a set of points in the phase plane such that the slope of vectors is constant. Geometrically, these are the points where the vectors all have the same slope. Algebraically, we find isoclines by solving $\frac{dy}{dx}=c$.

Jacobian matrix: A matrix that consists of all the first order partial derivatives of the differential equations in a system. When these partial derivatives are evaluated at a equilibrium solution, the Jacobian matrix linearizes a nonlinear system.

Linear system of differential equations: A system in which the dependent variables appear in linear combinations, that is, they may be multiplied only by scalar quantities and combined only through addition and subtraction. For example, a two dimensional first order linear system of differential equations can be written as follows, where a, b, c, and d are real numbers:

$$\frac{dx}{dt}=ax+by$$
$$\frac{dy}{dt}=cx+dy$$

Linearization: The linearization, $L(h)$, of a function around a point of interest, x^*, is given by $L(h)\equiv f(x^*)+hf'(x^*)$. The key feature of the linearization is that, when $x\approx x^*$, that is, $x=x^*+h$ for $h\approx 0$, then $f(x)\approx L(h)$.

Method of undetermined coefficients: This is a 3-step strategy to solve second order differential equations (1 - Find the general solution to the corresponding homogeneous equation; 2 - Find the particular solution to the nonhomogeneous equation, 3 - Add the previous results).

Nonhomogenous differential equation: A nonhomogeneous second order linear differential equation with constant coefficients has the form $y''+py'+q=g(t)$, where $g(t)$ is nonzero. More generally, $P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=G(t)$ is a second order linear differential equation, where $G(t)$ is not zero. The same holds true for higher order differential equations.

Nullcline: The x-nullcline is a set of points in the phase plane such that $\frac{dx}{dt} = 0$. Geometrically, these are the points where the vectors point either straight up or straight down. Algebraically, we find the x-nullcline by solving $\frac{dx}{dt} = 0$. The y-nullcline is a set of points in the phase plane so that $\frac{dy}{dt} = 0$. Geometrically, these are the points where the vectors are horizontal, pointing either to the left or to the right. Algebraically, we find the y-nullcline by solving $\frac{dy}{dt} = 0$. The x-nullcline and y-nullcline are specific **isoclines**.

Phase plane: A plane where solutions and/or vectors for as system of differential equations can be represented in two dimensions. You often will see vectors and/or solutions represented.

Phase portrait: Projection of the solution curves of a system like:

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

into the $x - y$ (phase) plane. Usually the phase portrait include several representative solutions to help represent all the solutions.

Vector field: A vector field shows a selection of vectors with the correct slope with normalized length in a phase plane.