MTH 267, Final Exam, Part I, Fall 2018 Name KEY

**Instructions**: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Consider the equation $\frac{d^2x}{dt^2} - x\cos(0.1x) = 0$. Convert the second-order equation to a first-order system. Linearize the equation and perform a stability analysis. Verify your solution using technology. (14 points)

$$\frac{d^2x}{dt^2} = x\cos(0.1x) \Rightarrow \frac{dx}{dt} = y \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}$$

$T = \frac{2\pi}{0.1}$

$T = 20\pi$

0's @ odd multiples of $5\pi$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & y \\ x\cos(0.1x) & 0 \end{pmatrix} \text{ linearize } \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

@ $5\pi$ $\begin{pmatrix} 0 & 1 \\ -\pi/2 & 0 \end{pmatrix}$ produces a stable orbit

Origin is a saddle point

$(-\lambda)(-\lambda) - 1 = 0$

$\lambda^2 - 1 = 0$

$(\lambda-1)(\lambda+1) = 0 \quad \lambda = \pm 1$

$\cos(0.1x) - x\sin(0.1x)(0.1)$

@ 0 $\Rightarrow 1$

2. A force of 10 lbs. stretches a spring 4 inches. A mass of 4 lbs. is attached to the end of the spring and is initially released from equilibrium position with a downward velocity of 2 in/s. Write the second-order equation that models the system. (7 points)

$10 = k(1/3)$

$k = 30$

$4 = 32m$

$\frac{1}{8} = m$

$$\frac{1}{8}y'' + 30y = 0$$

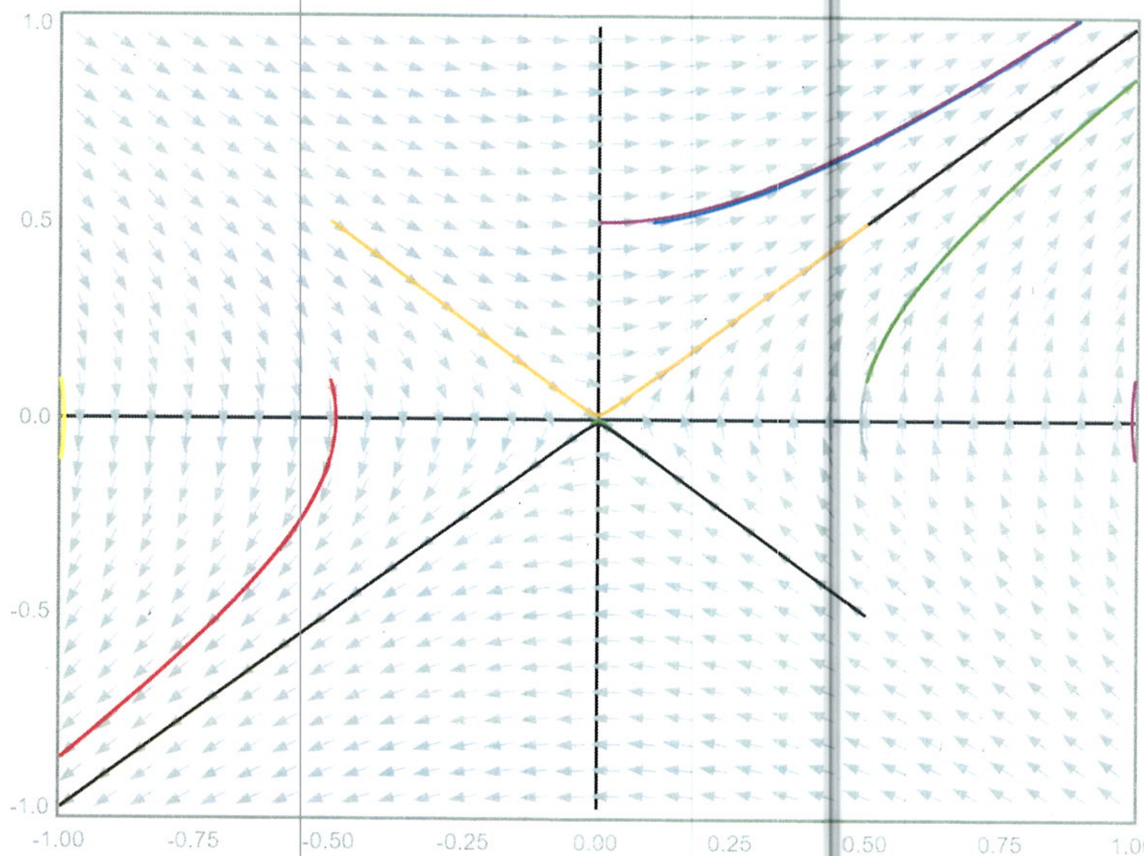
$y(0) = 0$

$y'(0) = -\frac{1}{6}$

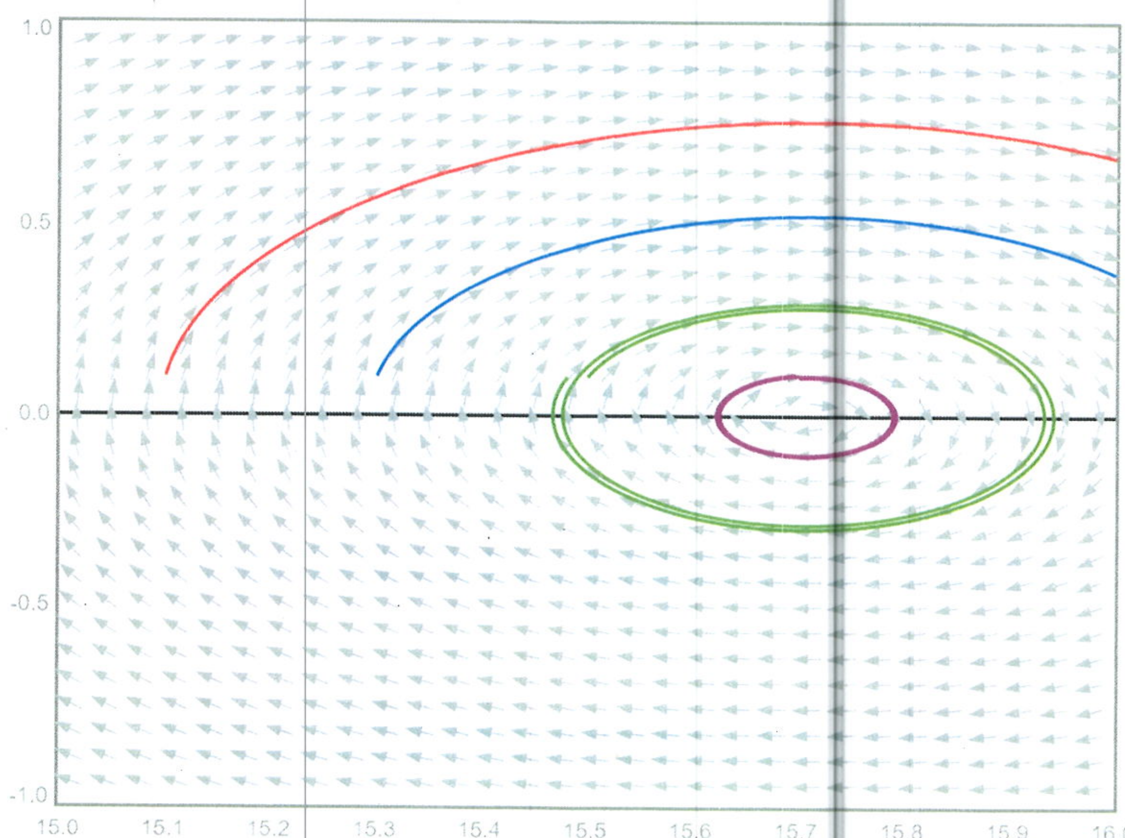
a. A force of $F(x) = \frac{1}{4}\cos(0.3x)$ is applied to the system. Write the new model. (5 points)

$$\frac{1}{8}y'' + 30y = \frac{1}{4}\cos(0.3x)$$

$$y'' + 240y = 2\cos(0.3x)$$



Origin



near $5\pi$

b. Solve the system using second-order methods, and the method of undetermined coefficients. (16 points)

$r^2 + 240 = 0$

$r = \pm\sqrt{240}\, i = \pm 4\sqrt{15}\, i$

$y(t) = C_1 \cos(4\sqrt{15}\,t) + C_2 \sin(4\sqrt{15}\,t) + 0.008336 \cos(0.3t)$

$Y_h = C_1 \cos(4\sqrt{15}\,t) + C_2 \sin(4\sqrt{15}\,t)$

$Y_p = A\cos(0.3t) + B\sin(0.3t)$

$Y_p' = -0.3A\sin(0.3t) + 0.3B\cos(0.3t)$

$Y_p'' = -0.09A\cos(0.3t) - 0.09B\sin(0.3t)$

$-0.09A\cos(0.3t) - 0.09B\sin(0.3t) + 240A\cos(0.3t) + B240\sin(0.3t) = 2\cos(0.3t)$

$(-0.09A + 240A)\cos(0.3t) = 2\cos(0.3t)$   $(-0.09B + 240B)\sin(0.3t) = 0$

$A = .008336$   $B = 0$

c. Sketch the graph of the solution for the given initial conditions. (8 points)

$0 = C_1 \cancel{\cos(4\sqrt{15}\cdot 0)}^{1} + C_2(0) + 0.008336\cancel{\cos(0.3\cdot 0)}^{1}$

$C_1 = -0.008336$

$y' = 0.008336\, 4\sqrt{15}\cancel{\sin(4\sqrt{15}\,t)}^{0} + 4\sqrt{15}\,C_2\cos(4\sqrt{15}\,t) + -0.008336(0.3)\cancel{\sin(0.3t)}^{0}$

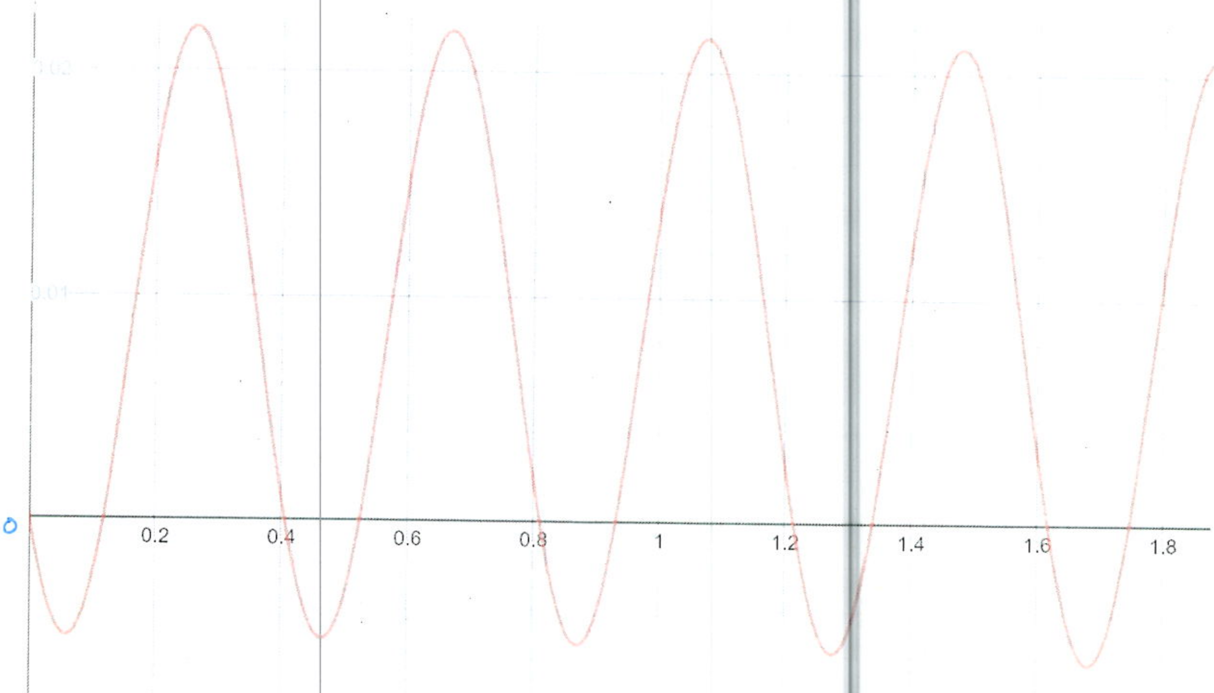
$-\frac{1}{6} = 4\sqrt{15}\,(1)\,C_1$

$C_1 = -0.01076$

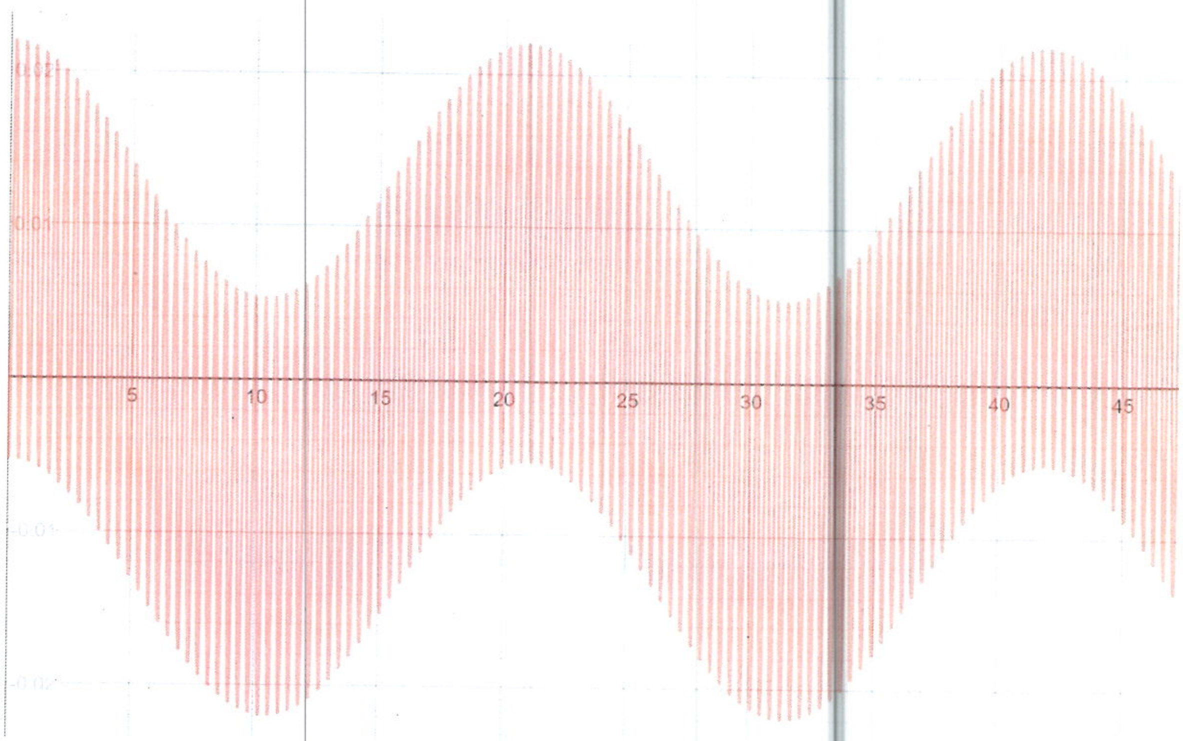
$y(t) = -0.008336\cos(4\sqrt{15}\,t) - 0.01076\sin(4\sqrt{15}\,t) + 0.008336\cos(0.3t)$

d. Does the system exhibit beats or resonance? Explain. (8 points)

no, since the period of the solution is not the same as the period of the forcing function



0
0.02
0.01
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

0.02
0.01
-0.01
-0.02
5
10
15
20
25
30
35
40
45


beats

3. Consider the competition model $\begin{cases} \frac{dx}{dt} = 0.75x + 0.25x^2 - xy \\ \frac{dy}{dt} = 0.5y - 0.1y^2 - xy \end{cases}$. Sketch the nullclines by hand for the system and use them to identify any equilibria. Using the information obtained from the nullclines, and a technology-generated full phase plane, can you characterize the equilibria as stable, unstable or a saddle point? Be sure to attach all your graphs. (15 points)

$0 = 0.75x + 0.25x^2 - xy = x(0.75 + 0.25x - y) = 0$

$x = 0 \qquad y = 0.25x + 0.75$

$0 = 0.5y - 0.1y^2 - xy = y(0.5 - 0.1y - x) = 0$

$y = 0 \qquad \frac{-0.1y}{-0.1} = \frac{x - 0.5}{-0.1} \Rightarrow y = -10x + 5$

critical points

$(0,0)$ unstable

$(0,5)$ stable

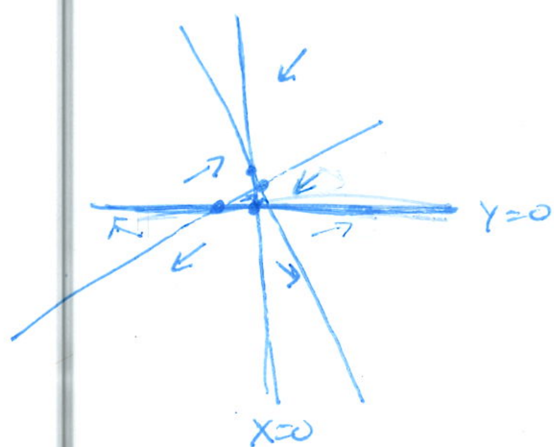
$(-3,0)$ unstable

$(0.415, 0.854) = \left(\frac{17}{41}, \frac{35}{41}\right)$ saddle

$0.25x + 0.75 = -10x + 5$

$10.25x = 4.25$

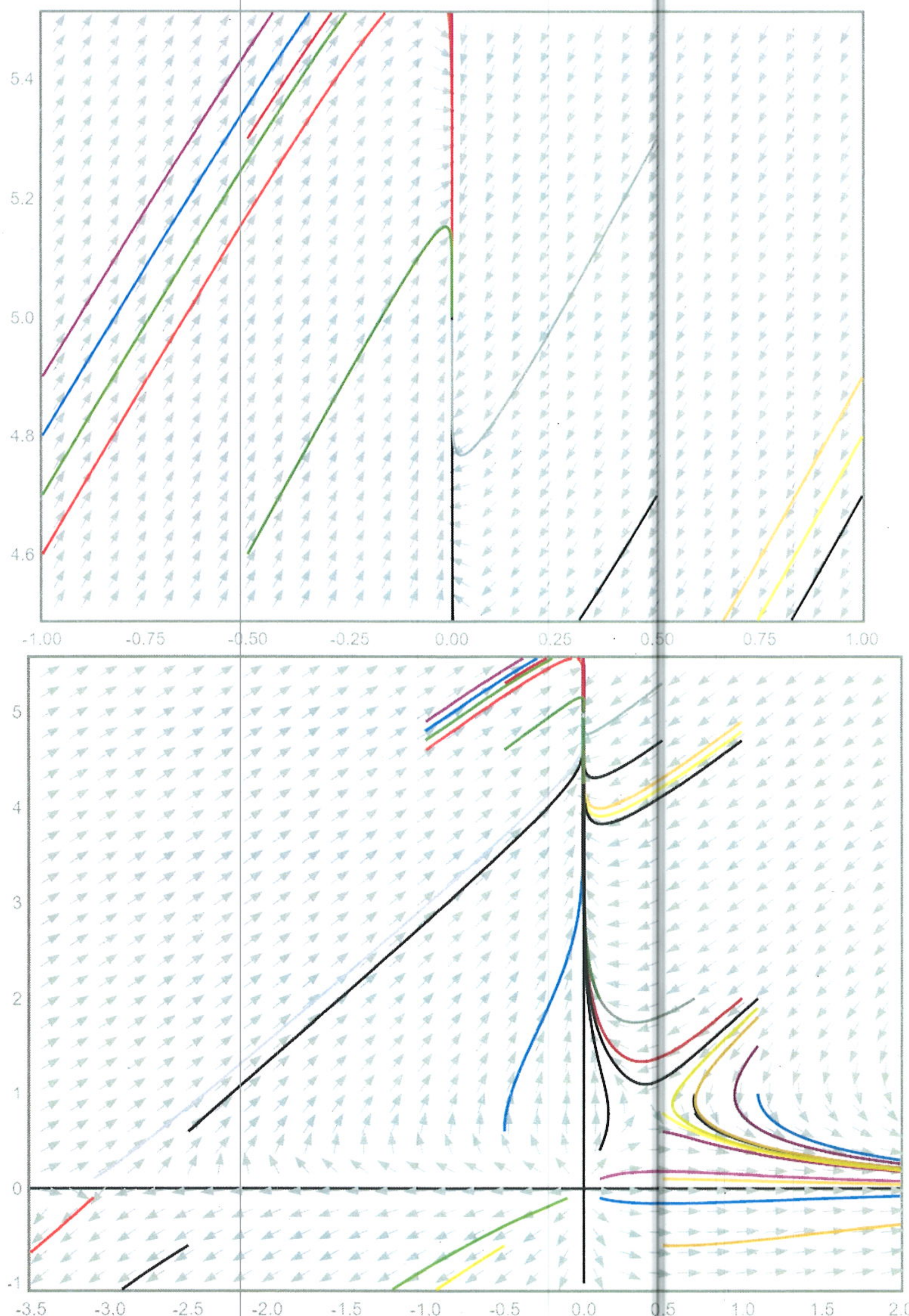
$x = \frac{17}{41}$



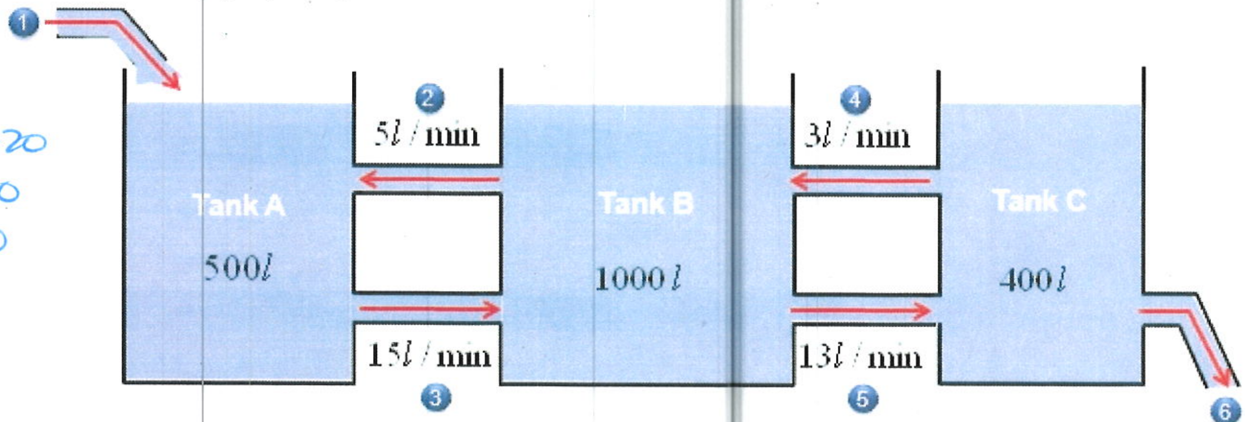

a. Does the system above model competition, cooperation or a predator-prey relationship? Explain your reasoning. (6 points)

Competition

both xy terms are negative


5.4
5.2
5.0
4.8
4.6
-1.00
-0.75
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00
5
4
3
2
1
0
-1
-3.5
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0

4. A three-tank system initially starts out with 20 kg of salt in tank A and none in tank B or Tank C. A 2 kg/L solution of brine is added to Tank A at a rate of 10 L/min. Water is cycled between the tanks as shown in the diagram, and then water flows out of Tank C at a rate of 10 L/min. Set up a system of differential equations that models the amount of salt in each tank. (You do not need to solve it.) (15 points)




$A(0)=20$
$B(0)=0$
$C(0)=0$

$$\frac{dA}{dt} = \frac{2\,kg}{L}\cdot\frac{10\,L}{min} - \frac{A}{500L}\cdot\frac{15L}{min} + \frac{B}{1000L}\cdot\frac{5L}{min} \qquad \frac{dA}{dt} = 20 - \frac{3A}{100} + \frac{B}{200}$$

$$\frac{dB}{dt} = \frac{3A}{100} - \frac{B}{1000L}\cdot\frac{18L}{min} + \frac{C}{400L}\cdot\frac{3L}{min} \qquad \frac{dB}{dt} = \frac{3A}{100} - \frac{9B}{500} + \frac{3C}{400}$$

$$\frac{dC}{dt} = \frac{B}{1000L}\cdot\frac{13L}{min} - \frac{3C}{400} - \frac{10C}{400} \qquad \frac{dC}{dt} = \frac{13B}{1000} - \frac{13C}{400}$$

5. Given the differential equation $\frac{dy}{dx} = 6\sin(y) - 0.3xy, y(0) = 3$, compute the value of $y(1)$ using $\Delta x = 0.05$ with Euler's Method. Illustrate two steps of the calculation by hand, and then complete the calculation using Excel. (25 points)

$X_0 = 0 \quad y_0 = 3 \quad m_0 = 6\sin(3) - 0.3(0)(3) = 0.846 \quad Y_1 = 0.846(0.05) + 3 = 3.04234$

$X_1 = 0.05 \quad y_1 = 3.04234 \quad m_1 = 6\sin(3.04234) - 0.3(0.05)(3.04234) = 0.5489$

$Y_2 = 0.5489(0.05) + 3.04234 = 3.06978$

$X_{20} = 1 \quad Y_{20} = 3.01478$

| Step (n) | t_n | y_n | m_n=f(t_n,y_n) | Delta_t=h | y_(n+1) |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 0.846720048 | 0.05 | 3.042336 |
| 1 | 0.05 | 3.042336 | 0.548927484 | 0.05 | 3.069782 |
| 2 | 0.1 | 3.069782 | 0.338397981 | 0.05 | 3.086702 |
| 3 | 0.15 | 3.086702 | 0.190275308 | 0.05 | 3.096216 |
| 4 | 0.2 | 3.096216 | 0.08639329 | 0.05 | 3.100536 |
| 5 | 0.25 | 3.100536 | 0.013732307 | 0.05 | 3.101222 |
| 6 | 0.3 | 3.101222 | -0.036953802 | 0.05 | 3.099375 |
| 7 | 0.35 | 3.099375 | -0.072201441 | 0.05 | 3.095765 |
| 8 | 0.4 | 3.095765 | -0.096619417 | 0.05 | 3.090934 |
| 9 | 0.45 | 3.090934 | -0.113451632 | 0.05 | 3.085261 |
| 10 | 0.5 | 3.085261 | -0.124977994 | 0.05 | 3.079012 |
| 11 | 0.55 | 3.079012 | -0.132798754 | 0.05 | 3.072372 |
| 12 | 0.6 | 3.072372 | -0.138035671 | 0.05 | 3.06547 |
| 13 | 0.65 | 3.06547 | -0.141474086 | 0.05 | 3.058397 |
| 14 | 0.7 | 3.058397 | -0.143663115 | 0.05 | 3.051214 |
| 15 | 0.75 | 3.051214 | -0.144986224 | 0.05 | 3.043964 |
| 16 | 0.8 | 3.043964 | -0.145710858 | 0.05 | 3.036679 |
| 17 | 0.85 | 3.036679 | -0.146023314 | 0.05 | 3.029378 |
| 18 | 0.9 | 3.029378 | -0.146053189 | 0.05 | 3.022075 |
| 19 | 0.95 | 3.022075 | -0.14589052 | 0.05 | 3.01478 |
| 20 | 1 | 3.01478 | -0.145597772 | 0.05 | 3.0075 |

| Step (n) | t_n | y_n | m_n=f(t_n,y_n) | Delta_t=h | y_(n+1) |
|---|---|---|---|---|---|
| 0 | 0 | 3 | =6*SIN(C2)-0.3*B2*C2 | 0.05 | =C2+D2*E2 |
| =A2+1 | =B2+E2 | =F2 | =6*SIN(C3)-0.3*B3*C3 | =E2 | =C3+D3*E3 |
| =A3+1 | =B3+E3 | =F3 | =6*SIN(C4)-0.3*B4*C4 | =E3 | =C4+D4*E4 |
| =A4+1 | =B4+E4 | =F4 | =6*SIN(C5)-0.3*B5*C5 | =E4 | =C5+D5*E5 |
| =A5+1 | =B5+E5 | =F5 | =6*SIN(C6)-0.3*B6*C6 | =E5 | =C6+D6*E6 |
| =A6+1 | =B6+E6 | =F6 | =6*SIN(C7)-0.3*B7*C7 | =E6 | =C7+D7*E7 |
| =A7+1 | =B7+E7 | =F7 | =6*SIN(C8)-0.3*B8*C8 | =E7 | =C8+D8*E8 |
| =A8+1 | =B8+E8 | =F8 | =6*SIN(C9)-0.3*B9*C9 | =E8 | =C9+D9*E9 |
| =A9+1 | =B9+E9 | =F9 | =6*SIN(C10)-0.3*B10*C10 | =E9 | =C10+D10*E10 |
| =A10+1 | =B10+E10 | =F10 | =6*SIN(C11)-0.3*B11*C11 | =E10 | =C11+D11*E11 |
| =A11+1 | =B11+E11 | =F11 | =6*SIN(C12)-0.3*B12*C12 | =E11 | =C12+D12*E12 |
| =A12+1 | =B12+E12 | =F12 | =6*SIN(C13)-0.3*B13*C13 | =E12 | =C13+D13*E13 |
| =A13+1 | =B13+E13 | =F13 | =6*SIN(C14)-0.3*B14*C14 | =E13 | =C14+D14*E14 |
| =A14+1 | =B14+E14 | =F14 | =6*SIN(C15)-0.3*B15*C15 | =E14 | =C15+D15*E15 |
| =A15+1 | =B15+E15 | =F15 | =6*SIN(C16)-0.3*B16*C16 | =E15 | =C16+D16*E16 |
| =A16+1 | =B16+E16 | =F16 | =6*SIN(C17)-0.3*B17*C17 | =E16 | =C17+D17*E17 |
| =A17+1 | =B17+E17 | =F17 | =6*SIN(C18)-0.3*B18*C18 | =E17 | =C18+D18*E18 |
| =A18+1 | =B18+E18 | =F18 | =6*SIN(C19)-0.3*B19*C19 | =E18 | =C19+D19*E19 |
| =A19+1 | =B19+E19 | =F19 | =6*SIN(C20)-0.3*B20*C20 | =E19 | =C20+D20*E20 |
| =A20+1 | =B20+E20 | =F20 | =6*SIN(C21)-0.3*B21*C21 | =E20 | =C21+D21*E21 |
| =A21+1 | =B21+E21 | =F21 | =6*SIN(C22)-0.3*B22*C22 | =E21 | =C22+D22*E22 |

MTH 267, Final Exam, Part II, Fall 2018 Name KEY

**Instructions**: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

6. Identify the Ansatz to find the particular solution for the differential equation $y'' + 4y' + 9y = f(x)$. (8 points each)

   a. $f(x) = 6\sin 4x - 5\cos 2x$

   $r^2+4r+9=0$ $\quad \frac{-4\pm\sqrt{16-36}}{2} = -2\pm\sqrt{5}i$

   $Y_p = A\sin 4x + B\cos 4x + C\sin 2x + D\cos 2x$

   b. $f(x) = 3e^{-x} + 2x - 1$

   $Y_p = Ae^{-x} + Bx + C$

   c. $f(x) = 9xe^{-x} + 14e^{-2x}\sin 2x$

   $Y_p = Axe^{-x} + Be^{-x} + Ce^{-2x}\sin 2x + De^{-2x}\cos 2x$

7. Solve the differential equation $\frac{dy}{dt} = \frac{t\sqrt{1-y^2}}{e^{2t}}$ using separation of variables. (16 points)

$$\int \frac{dy}{\sqrt{1-y^2}} = \int te^{-2t}\,dt$$

$$\arcsin y = -\tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} + C$$

$$Y = \sin\left[-\tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} + C\right]$$

8. Solve the differential equation $y' + xy = x, y(1) = 3$ using the method of integrating factors (reverse product rule). (16 points)

$$\mu = e^{\int x\,dx} = e^{\frac{1}{2}x^2}$$

$$e^{\frac{1}{2}x^2} y' + x e^{\frac{1}{2}x^2} y = x e^{\frac{1}{2}x^2}$$

$$\int \left(e^{\frac{1}{2}x^2} y\right)' = \int x e^{\frac{1}{2}x^2}$$

$$e^{\frac{1}{2}x^2} y = e^{\frac{1}{2}x^2} + C$$

$$y = 1 + Ce^{-\frac{1}{2}x^2}$$

9. Solve the systems of equations for the general solution below using eigenvalues. Be sure that your solutions are expressed only with real-valued functions. Sketch sample trajectories in the phase plane and determine the character of the origin (attractor, repeller, saddle point). (20 points)

$$\vec{x}'(t) = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} \vec{x}$$

$$(1-\lambda)(4-\lambda) - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

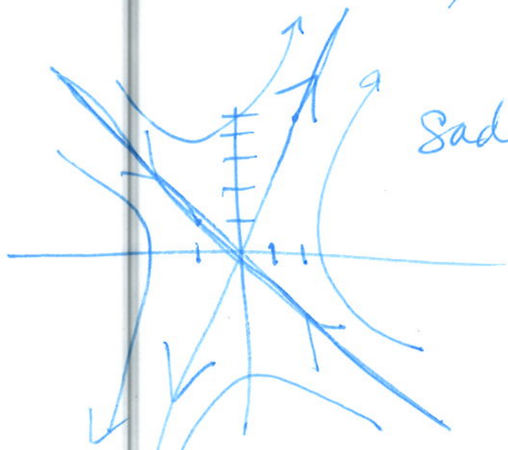
$$\lambda = 6, -1$$

$\lambda = 6$ $\begin{pmatrix} -5 & 2 \\ 5 & -2 \end{pmatrix}$ $5x_1 = 2x_2$, $x_1 = \frac{2}{5}x_2$ $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\lambda = -1$

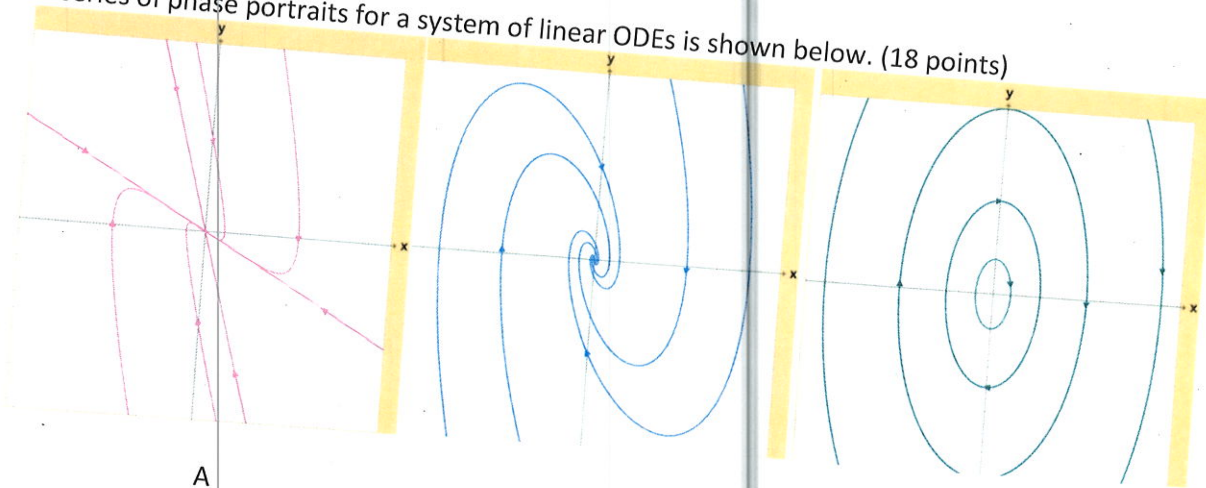
$\begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}$ $x_1 = -x_2$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$



Saddle

10. A series of phase portraits for a system of linear ODEs is shown below. (18 points)




A B C

Match the phase portrait with one of the differentia equations below that could be represented by the graph. Explain your reasoning.

a. $y'' + 3y = 0$

C sine/cosine/pure imaginary solution
only stable orbit
undamped

b. $y'' + 4y' + 3y = 0$

$(r+3)(r+1)$

$r=-3,-1$

A attractor, exponentially
overdamped

c. $y'' + 2y' + 5y = 0$

$\frac{-2 \pm \sqrt{4-20}}{2}$

B complex solutions, negative real
spirals in
underdamped

11. Solve the second order differential equation $y'' + 4y' + 4y = 0$ for the general solution. (14 points)

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

12. Use the Existence and Uniqueness Theorem to determine where the differential equation $y' = \frac{2\sqrt{1-y^2}}{x^2}$ is guaranteed to have a unique solution. Sketch the graph of the region. (14 points)

$x^2 \neq 0$

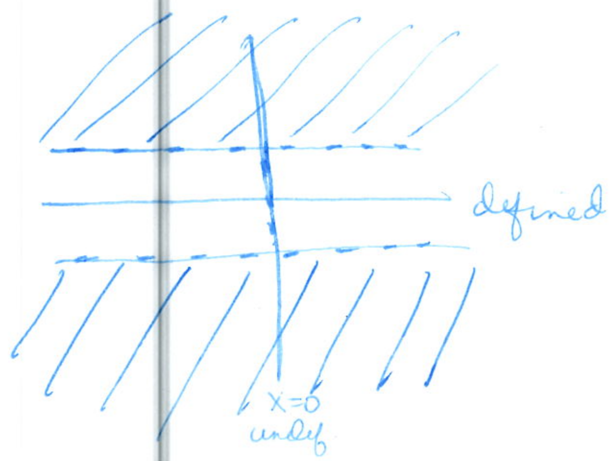
$1-y^2 \geq 0$

$1 \geq y^2 \qquad -1 \leq y \leq 1$

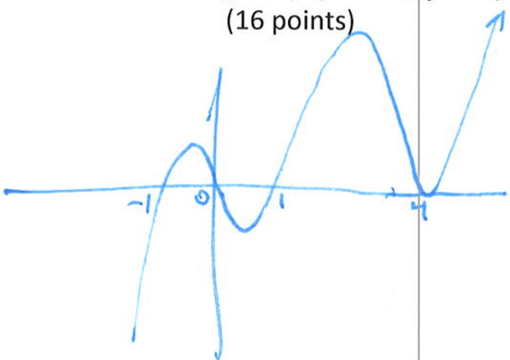
$f = \dfrac{2(1-y^2)^{1/2}}{x^2}$

$f_y = \frac{2}{x^2}\cdot\frac{1}{2}(1-y^2)^{-1/2}(-2y) = \dfrac{-2y}{x^2\sqrt{1-y^2}}$

$1-y^2 \neq 0 \Rightarrow y \neq \pm 1$



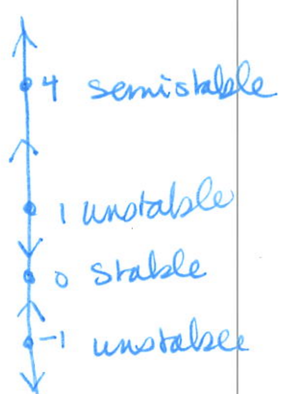

13. Draw a phase line and phase plane (graph of $y'$ vs $y$) for the autonomous differential equation $y' = y(y^2-1)(y-4)^2$. Characterize each equilibrium as stable, unstable or semi-stable. (16 points)



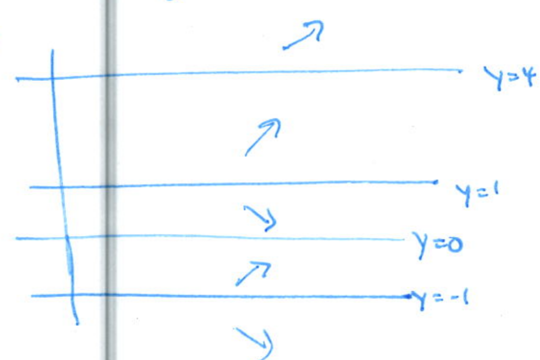

phaseplane

slope field

phase line




$\Rightarrow$



14. *Direction fields for population models are shown below. (18 points)*

a. *Find a differential equation that models the population (up to a constant multiple).*

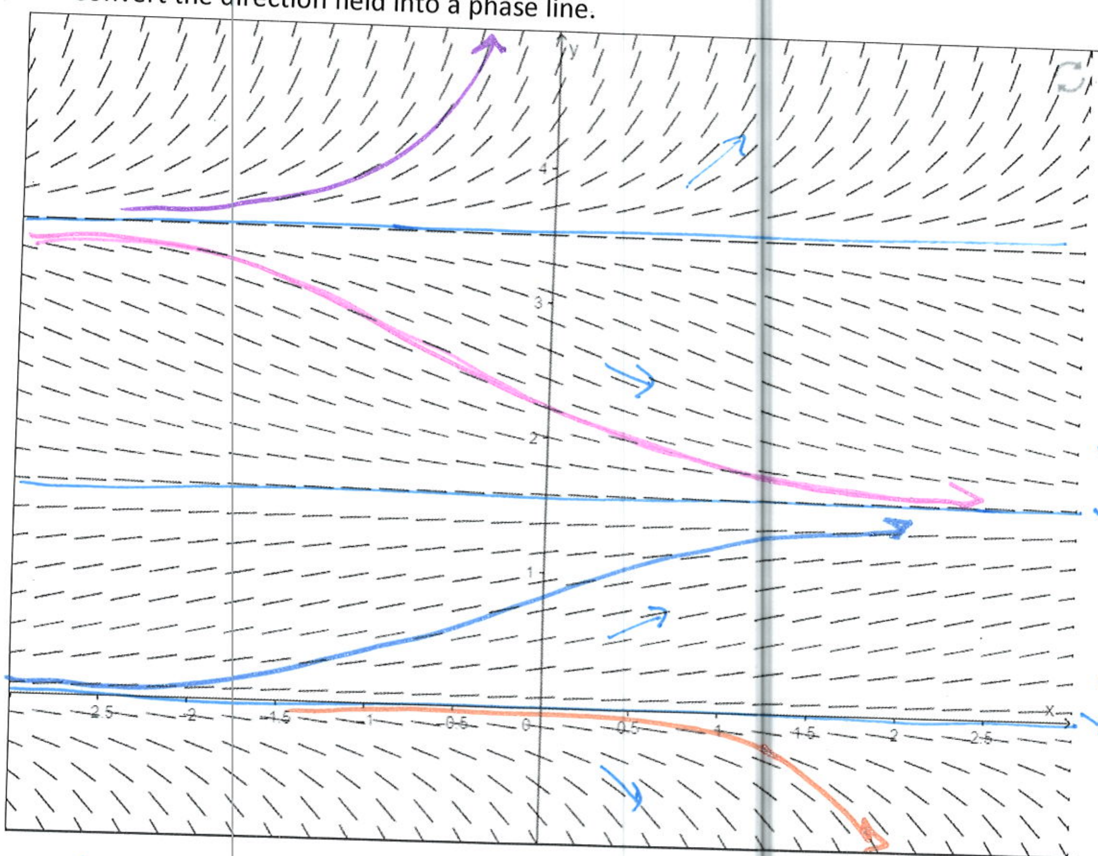
b. Plot trajectories of initial conditions that models each type of trajectory for the model.

c. Where is the model logistic?

d. Describe the long-term behavior of each trajectory.

e. Describe each equilibrium as a carrying capacity or a threshold.

f. Convert the direction field into a phase line.




$\frac{dy}{dt} = y\left(y-\frac{5}{2}\right)\left(y-\frac{7}{2}\right)$

$y=\frac{7}{2}$ threshold

$y=\frac{5}{2}$ carrying capacity