MTH 267, Exam #1, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Use technology to graph the direction/slope field for each differential equation below. For each slope field, select two different sets of initial conditions and plot the trajectory of the graph from that point. Note any equilibria (points or lines where the slope is zero). Attach your graphs. (10 points each)
   a. $\frac{dy}{dx} = \frac{x(y^2+x)}{5}$
   b. $\frac{dy}{dx} = \frac{\ln(x^2+1)y(x^2-4)}{4}$

2. Solve the homogeneous equation $4(x^2+y^2)dx + xydy = 0$. (10 points)

homogeneous degree 2

$$\cancel{xy}\frac{dy}{dx} = \frac{-4(x^2+y^2)\cancel{dx}}{xy}$$

let $y = vx$

$\frac{dy}{dx} = y' = v'x + v$

$$v'x + v = \frac{-4(x^2+v^2x^2)}{x \cdot vx} = \frac{-4\cancel{x^2}(1+v^2)}{\cancel{x^2}v}$$

$$v'x + \cancel{v} = \frac{-4(1+v^2)}{v} - v \cdot \frac{v}{v}$$

$$v'x = \frac{-4-4v^2-v^2}{v} = \frac{-4-5v^2}{v}$$

$u = 4+5v^2$

$du = 10v\,dv$

$$\int \frac{v}{4+5v^2}dv = \int -\frac{4}{x}dx$$

$$\frac{1}{10}\ln|4+5v^2| = -4\ln x + C$$

$$\cancel{\ln}|(4+5v^2)^{1/10}| = \cancel{\ln}x^{-4} + C = \ln(Ax^{-4})$$

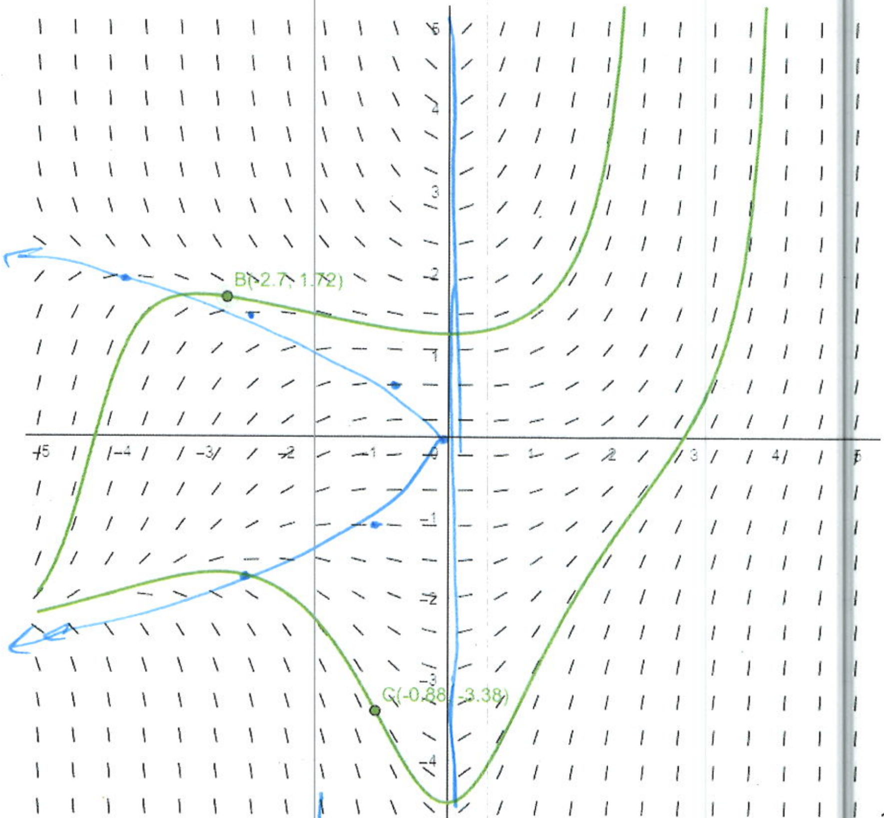
$$\left[(4+5v^2)^{1/10} = Ax^{-4}\right]^{10}$$

$v = y/x$

$$4+5v^2 = Ax^{-40}$$

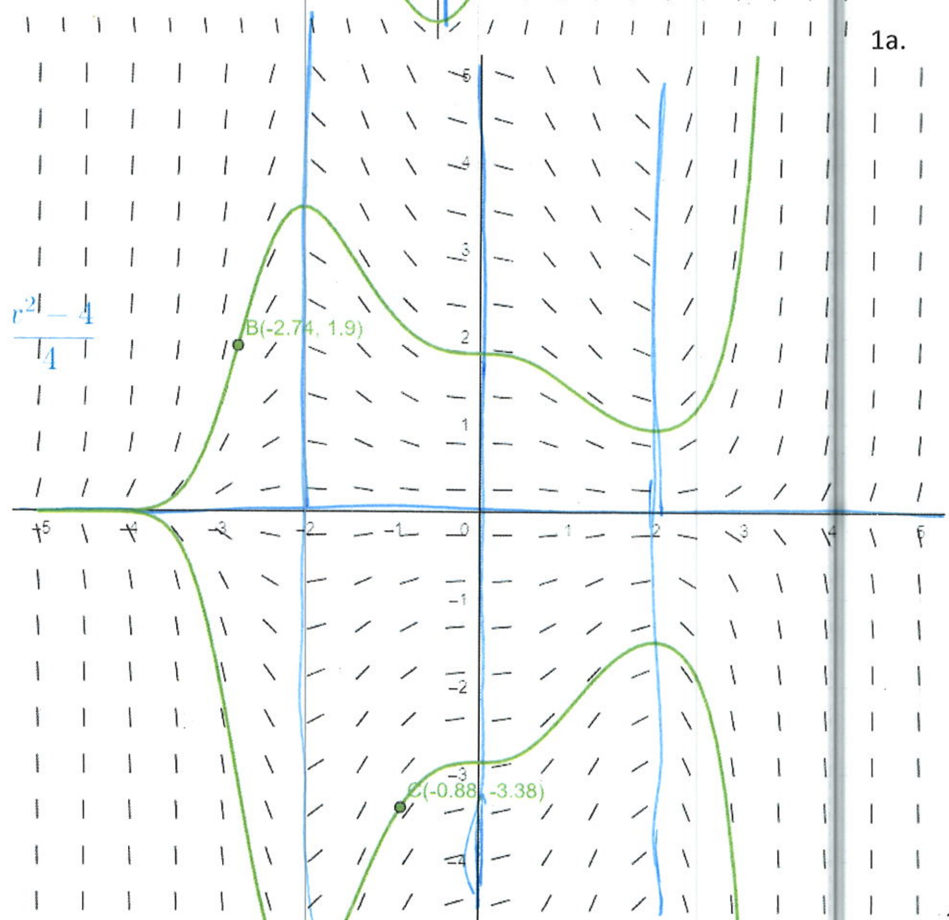
$$\cancel{5}v^2 = \frac{Ax^{-40}-4}{5}$$

$$\frac{y^2}{x^2} = \frac{A}{5}x^{-40} - \frac{4}{5} \Rightarrow y^2 = \frac{A}{5}x^{-38} - \frac{4}{5}x^2$$



$\frac{dy}{dx} = 0$ when $x = 0$ and $x = -y^2$

1a.



$\frac{dy}{dx} = 0$ when

$x = 0$

$y = 0$

$x = \pm 2$

1b

3. Solve the Bernoulli equation $xy' + 4y = x^4y^3$. (10 points)

$y' + \frac{4}{x}y = x^3y^3$ $\quad n=3 \quad (1-n)y^{-n} = -2y^{-3}$

$-2y^{-3}y' + \frac{4}{x}(-2y^{-2}) = -2x^3$ $\quad z = y^{-2}$

$z' - \frac{8}{x}z = -2x^3$ $\quad z' = -2y^{-3}y'$

$x^{-8}z' - 8x^{-9}z = -2x^{-5}$ $\quad \mu = e^{\int -\frac{8}{x}dx} = e^{-8\ln x} = e^{\ln x^{-8}} = x^{-8}$

$\int (x^{-8}z)' = \int -2x^{-5}dx$

$x^{-8}z = \frac{-2}{-4}x^{-4} + C$ $\quad y^2 = \frac{2x^{-4}}{1+cx^4} = \frac{2}{x^4(1+Cx^4)}$

$z = \frac{1}{2}x^{-4}x^8 + cx^8$

$z = \frac{1}{2}x^4 + Cx^8$

$\frac{1}{y^2} = y^{-2} = \frac{1}{2}x^4 + Cx^8 = \frac{1}{2}x^4(1 + Cx^4)$

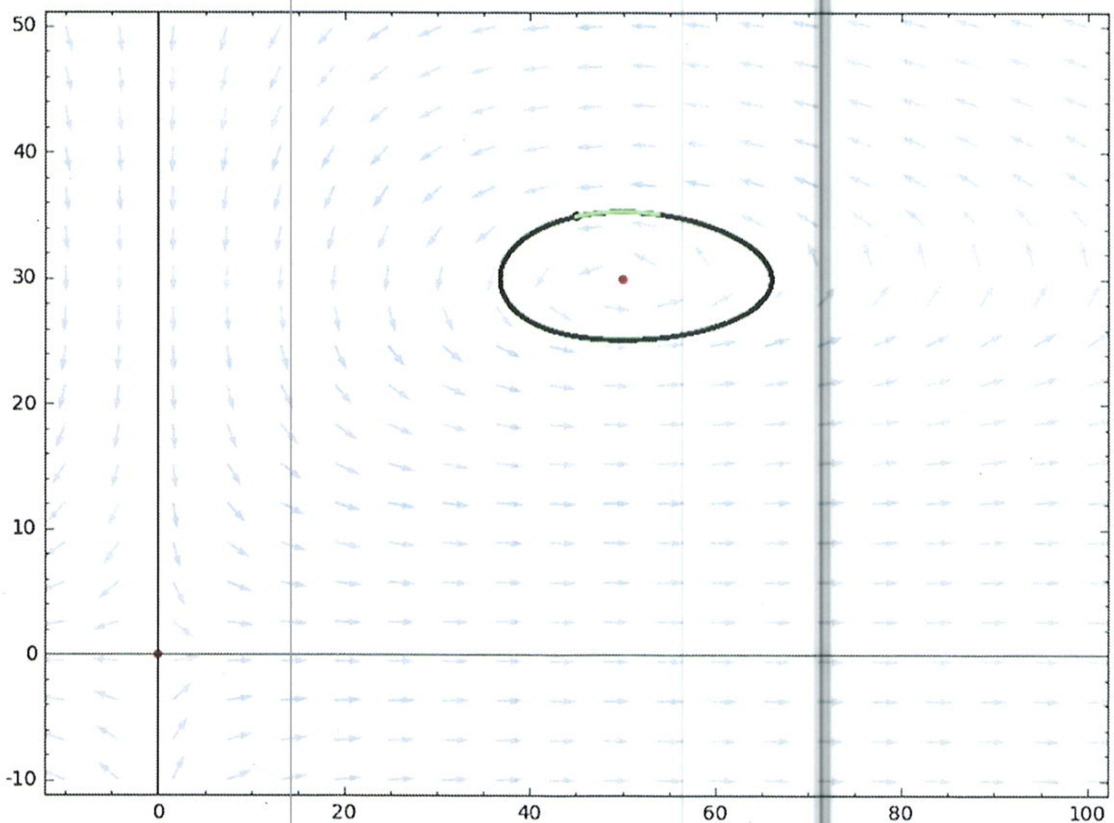
4. Consider the system of differential equations $\begin{cases} \frac{dx}{dt} = 0.6x - 0.02xy \\ \frac{dy}{dt} = -0.2y + 0.004xy \end{cases}$. Use technology to graph the phase plane. Describe the behavior of the system. Identify any equilibria and characterize its stability. Describe the behavior of each population in the absence of the other one. Determine if the population model is cooperative, competitive or predator-prey. Explain your reasoning. (12 points)

$0 = 0.6x - 0.02xy = x(.6 - .02y)$ $\quad x=0 \quad y=30$

$0 = -0.2y + 0.004xy = -y(0.2 - .004x)$ $\quad y=0 \quad x=50$

equilibria at $(0,0)$ and $(30,50)$

Saddle Stable



50
40
30
20
10
0
-10
0
20
40
60
80
100

MTH 267, Exam #1, Part II, Fall 2018 Name ____________________

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5. Verify that $y = -\sqrt{\ln^2 x + e}$ is a solution to the differential equation $\frac{dy}{dx} = \frac{\ln x}{xy}$. [Do not solve the equation from scratch. Verify the given solution. No integration is involved.] (10 points)

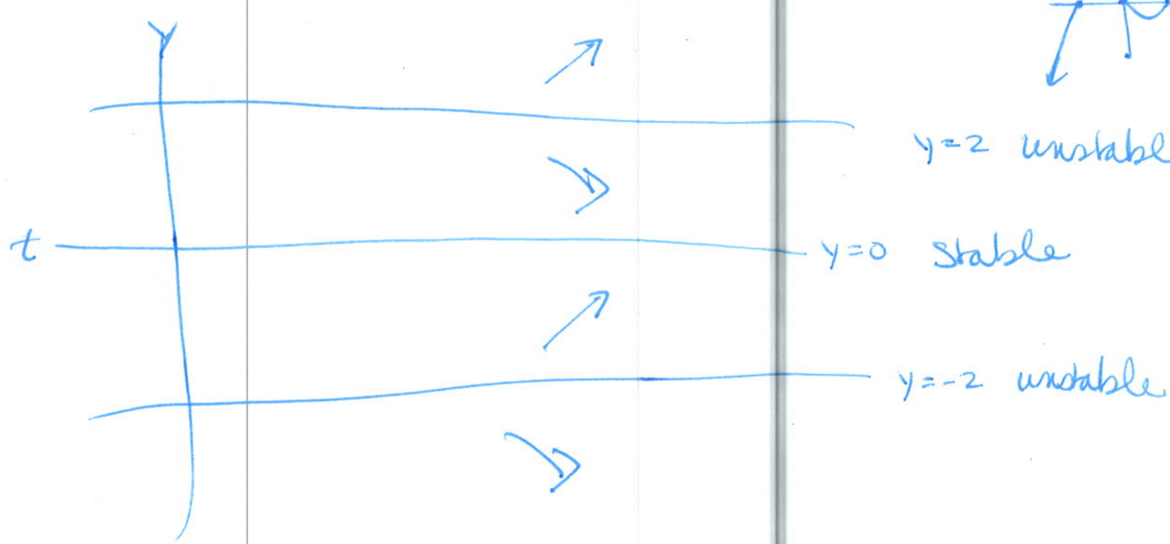
$$y' = -\frac{1}{\cancel{2}}(\ln^2 x + e)^{-1/2} \cdot \cancel{2}\ln x \cdot \frac{1}{x}$$

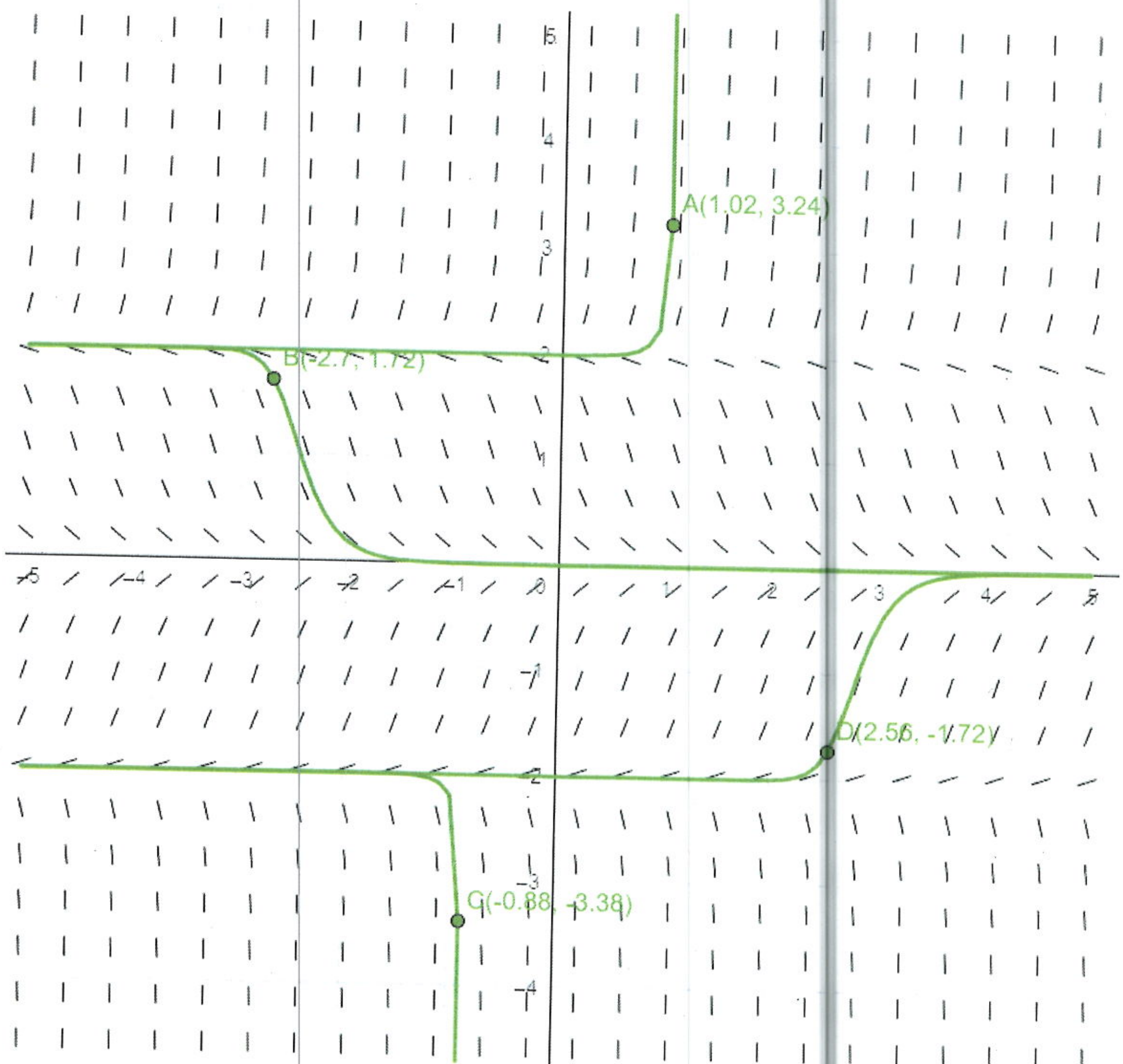
$$= \frac{-\ln x}{x\left(-\sqrt{\ln^2 x + e}\right)} = \frac{-\ln x}{xy}$$

6. Graph the direction field for the equation $y' = y(y^2 - 4)$ by hand. Note any equilibria and describe their stability. (10 points)

$$0 = y(y^2-4) = y(y-2)(y+2)$$

equilibria at $y=0, y=\pm 2$




A(1.02, 3.24)
B(-2.7, 1.72)
C(-0.88, -3.38)
D(2.56, -1.72)

7. Use the properties of the Uniqueness and Existence Theorem to determine where the differential equation $y' = \frac{\ln(x^2-1)}{\sqrt{4-x^2-y^2}}$ is guaranteed to have a solution. Sketch the graph of the restrictions. (12 points)

$$f = \frac{\ln(x^2-1)}{\sqrt{4-x^2-y^2}}$$

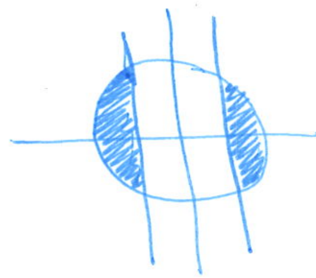
$4 - x^2 - y^2 > 0$

$4 > x^2 + y^2$

inside circle of radius 2

$\ln(x^2-1) \Rightarrow x^2 - 1 > 0$

$(-\infty,-1) \cup (1,\infty)$ defined



Shaded region is defined

$$f_y = \frac{\ln(x^2-1)\left(-\frac{1}{2}\right)(-2y)}{\left(\sqrt{4-x^2-y^2}\right)^3}$$

defined in same places

8. Use Euler's Method with the indicated step size to find an estimate of the solution at the indicated point: $y' = \frac{1}{4}xy - 1, y(2) = 5, \Delta x = 0.5, y(3) = ?$ (15 points)

$x_0 = 2,\ y_0 = 5 \quad m_0 = \frac{1}{4}(2)(5) - 1 = 1.5 \quad y_1 = 0.5(1.5) + 5 = 5.75$

$x_1 = 2.5 \quad y_1 = 5.75 \quad m_1 = \frac{1}{2}(2.5)(5.75) - 1 = 2.59 \quad y_2 = 2.59(0.5) + 5.75 = 7.05$

$\boxed{x_2 = 3 \quad y_2 = 7.05}$

9. Solve the differential equation $\frac{dy}{dx} = \frac{y^2+1}{xy}, y(1) = 2$ using separation of variables. (12 points)

$$\int \frac{y}{y^2+1}\,dy = \int \frac{1}{x}\,dx$$

positive root

$$\frac{1}{2}\ln|y^2+1| = \ln \cancel{x} + C$$

$$\ln|y^2+1| = \ln x^2 + C = \ln(Ax^2)$$

$$y^2+1 = Ax^2$$

$$y^2 = Ax^2 - 1$$

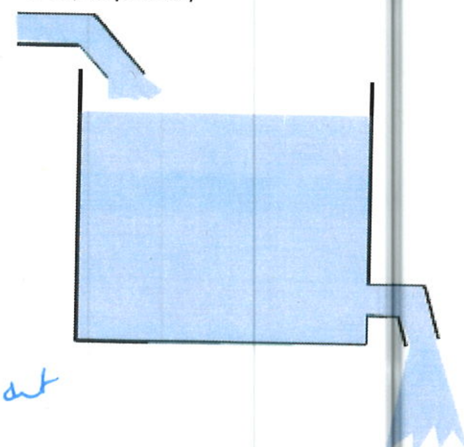
$$y = \sqrt{Ax^2-1}$$

$$2 = \sqrt{A(1)^2 - 1}$$

$$4 = A - 1 \Rightarrow A = 5$$

$$y = \sqrt{5x^2-1}$$

10. Initially a tank contains 1kg of salt dissolved in 1000L of water. Salty water containing 4kg/L at a rate of 3L/min is added to the tank and the (stirred) solution is draining from the tank at 4L/min. Determine an equation for how much salt is in the tank at any time t. Determine the amount of salt in the tank when 500L is left. (15 points)



$$A(0) = 1\,\text{kg}$$

$$\frac{dA}{dt} = \text{Rate}_{in} - \text{Rate}_{out}$$

$$\frac{dA}{dt} = \frac{4\,\text{kg}}{\cancel{L}} \cdot \frac{3\cancel{L}}{m} - \frac{A}{1000\cancel{L} - t} \cdot \frac{4\cancel{L}}{min}$$

$$\mu = e^{\int \frac{4}{1000-t}dt}$$

$$= -4\ln(1000-t)$$

$$e$$

$$= (1000-t)^{-4}$$

$$\frac{dA}{dt} = 12 - \frac{4A}{1000-t} \Rightarrow A' + \frac{4}{1000-t}A = 12$$

$$(1000-t)^{-4}A' + 4(1000-t)^{-5}A = 12(1000-t)^{-4}$$

$$\int \left((1000-t)^{-4}A\right)' = \int 12(1000-t)^{-4}dt$$

$$(1000-t)^{-4}A = \frac{12(-1)}{-3}(1000-t)^{-3} + C$$

$$A = 4(1000-t) + C(1000-t)^4$$

$$1 = 4000 + C(1000-t)^4$$

$$C = -3.999 \times 10^{-9}$$

$$A(t) = 4000 - 4t - 3.999\times10^{-9}(1000-t)^4$$

500L left at $t = 500$ $\Rightarrow A(500) = 1750$ kg

11. Solve the linear differential equation $y' - 3y = -3x$, by the method of integrating factors (reverse product rule). (12 points)

$$\mu = e^{\int -3dx} = e^{-3x}$$

$$e^{-3x}y' - 3e^{-3x}y = -3xe^{-3x}$$

$$(e^{-3x}y)' = \int -3xe^{-3x}dx$$

$$e^{-3x}y = \frac{1}{3}(3x+1)e^{-3x} + C$$

$$y = \frac{1}{3}(3x+1) + Ce^{3x}$$

12. Classify the differential equations by order, linearity and whether it is ordinary or partial. (2 points each)

a. $\frac{dy}{dt} = t^2y - \cos t$ 1st linear ordinary

b. $\left(\frac{dy}{dy}\right)^2 = \ln t + y$ 1st nonlinear ordinary

c. $u_{xx} - u_{yy} = u_{xy}$ 2nd linear partial